Math 350 – Groups, Rings and Fields – Homework 4 (Section 7 and start of 12)

Due: Friday March 8th at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. Give an example of a bijection from $\mathbb{R}$ to the open interval $(0, 1)$.
2. Let $f: S \to T$ and $g: T \to U$ be two functions.
   (a) Suppose that $g \circ f$ is injective. Do f and g need to both be injective? Why or why not?
   (b) Suppose that $g \circ f$ is surjective. Do f and g need to both be surjective? Why or why not?
3. Let G be a group, and let $h \in G$ be an element. Define a function $a_h: G \to G$ by $a_h(g) = hgh^{-1}$. Prove that a_h is a bijection.
4. Let G be an abelian group, choose $n \in \mathbb{Z}_{>0}$ and define a function $\phi_n: G \to G$ by $\phi_n(g) = g^n$.
   (a) Prove that ϕ_n is a homomorphism.
   (b) Give an example to show that ϕ_n does not need to be either injective or surjective.

B

5. Let V be the Klein four group. List all the automorphisms of V (in other words, all the isomorphisms $\phi: V \to V$). Explain why your list is complete.
6. Let $\phi: D_n \to \mathbb{Z}/n$ be a homomorphism. (Reminder: D_n is the group of symmetries of a regular n-gon. It has $2n$ elements, namely r^m and sr^m where $0 \leq m < n$. The element r corresponds to a rotation by $2\pi/n$, and $r^n = e$. The element s corresponds to reflection in the x-axis, and $s^2 = e$. Finally, we have $sr = r^{-1}s$.)
   (a) If n is odd, prove that ϕ is the trivial homomorphism, i.e. that $\phi(g) = 0$ for all $g \in D_n$ (Hint: first show $\phi(s)$ has order ≤ 2, then use $sr = r^{-1}s$ to show that $\phi(r)$ has order ≤ 2).
   (b) What can you say about the homomorphism ϕ if n is even? (Use the hint from part (a) again and think about the order 2 elements of $\mathbb{Z}/n$).
7. $(*)$ (*Optional*.) Let G be the group of *isometries* of the complex plane $\mathbb{C}$. In other words, G consists of all maps $A: \mathbb{C} \to \mathbb{C}$ where either $A(z) = az + b$, or $A(z) = a\overline{z} + b$, where a and b are complex numbers, $|a| = 1$, and $\overline{z}$ is the complex conjugate of z (that is, $\overline{x + iy} = x - iy$).
   (a) Describe all finite order elements of the group G.

(b) Describe all finite subgroups H of the group G. Hint: see what the elements of H do to the following point $C_H \in \mathbb{C}$:

$$C_H = \frac{1}{|H|} \sum_{h \in H} h(0).$$