Math 350 – Groups, Rings and Fields – Homework 5 (Start of §12 and §8)

Due: Friday March 15th at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. In each case determine whether or not the groups are isomorphic.
   (a) $\mathbb{Z}/3 \times \mathbb{Z}/3$ and $\mathbb{Z}/9$.
   (b) $D_3 \times \mathbb{Z}/4$ and $D_4 \times \mathbb{Z}/3$.
   (c) D_3 and S_3.
2. If G is isomorphic to H prove that the center $Z(G)$ is isomorphic to the center $Z(H)$. (We defined the center in Homework 1 Question 5).
3. Write the following elements of S_n as products of disjoint cycles. Compute the parity and the order of each element.
   (a) $(1\,2)(4\,3\,2\,1)(1\,2) \in S_4$.
   (b) $(3\,4\,5)(1\,2)(3\,9\,8\,7\,6)(2\,3) \in S_9$.
4. (a) Give an example of an element of the group S_{10} of order 14.
   (b) Prove that there are no elements of the group A_{10} of order 14.

B

5. Let $H \subseteq S_n$ be any subgroup. Prove that *either* $H \cap A_n = H$ (i.e. all the elements of H are even), *or* $|H|$ is even and $|H \cap A_n| = \frac{1}{2}|H|$ (i.e. exactly half of the elements of H are even).
6. If G is a group, let $\mathrm{Aut}(G)$ be the set of isomorphisms $\phi\colon G \to G$.
   (a) Prove that $\mathrm{Aut}(G)$ is a subgroup of S_G.
   (b) To what familiar group is $\mathrm{Aut}(\mathbb{Z})$ isomorphic? Prove your answer.
   (c) To what familiar group is $\mathrm{Aut}(\mathbb{Z}/12)$ isomorphic? Prove your answer.
7. $(*)$ (*Optional.*) Consider a deck of $2n$ cards. An *outer Faro shuffle* is a way of shuffling the deck of cards: you first cut the deck into two equal stacks of n cards, then interleave the cards, alternating between the two stacks, so that the bottom and top cards of the deck end up in the same positions as they started. View the outer Faro shuffle as an element $F \in S_{2n}$. Prove that $F^k = e$ if and only if $2^k = 1 \bmod 2n - 1$. Compute the order of F if $2n = 52$.