Math 350 – Groups, Rings and Fields – Homework 7 (§16 and start of 17)

Due: Friday April 5th at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. If R_1 and R_2 are rings, define a ring structure on the product set $R_1 \times R_2$ by $(r_1, r_2)+(s_1, s_2) = (r_1+s_1, r_2+s_2)$ and $(r_1, r_2) \times (s_1, s_2) = (r_1 s_1, r_2 s_2)$. Prove that this makes $R_1 \times R_2$ into a ring. We will write $R_1 \oplus R_2$ for this ring and call it the *direct sum* of R_1 and R_2.
2. Show that the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \colon a, b \in \mathbb{Q}\}$ forms a field under the usual operations of addition and multiplication.
3. Find all units, zero divisors and nilpotent elements in each of the following rings.
   (a) $\mathbb{Z} \oplus \mathbb{Z}$
   (b) $\mathbb{Z}/3 \oplus \mathbb{Z}/3$
   (c) $\mathbb{Z}/4 \oplus \mathbb{Z}/6$.
4. Let S, T be subrings of a ring R.
   (a) Prove that $S \cap T$ is a subring of R.
   (b) Prove that $S \cup T$ is a subring of R if and only if $S \subseteq T$ or $T \subseteq S$.

B

5. If R is any ring, let $G = \langle 1 \rangle$ be the cyclic subgroup generated by the multiplicative identity under the operation $+$. So $G \cong \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 0$. We denote the integer n by $\mathrm{char}(R)$, and call it the *characteristic* of R.
   (a) Prove that is R is an integral domain then $\mathrm{char}(R)$ is either zero or a prime number.
   (b) Prove that if R is a commutative ring of prime characteristic p then $(x + y)^p = x^p + y^p$ for all $x, y \in R$. (This is sometimes called the *Freshman's dream*).
6. Let X be a set and let $R = \mathbb{P}(X)$. Define operations $+, \times$ on R by

$$U + V = U \Delta V, \quad U \times V = U \cap V,$$

   so multiplication is given by intersection, and addition by the symmetric difference operator (if you forgot the definition look at homework 1 problem 4).
   (a) Prove that these operations make R into a commutative ring in which $U^2 = U$ for all $U \in R$.

(b) Prove that if R is *any* ring in which $a^2 = a$ for all $a \in R$ then $\mathrm{char}(R) = 2$.

7. $(*)$ *Optional:* Let π be a set of prime numbers. Let $\mathbb{Z}_\pi \subseteq \mathbb{Q}$ be the subset of the rational numbers consisting of rationals a/b where the only prime factors of b are in the set π. Show that a subset $S \subseteq \mathbb{Q}$ is a subring if and only if $S = \mathbb{Z}_\pi$ for some set π of prime numbers.