# Math 350 – Groups, Rings and Fields – Homework 8 (§17 and 18)

**Due: Friday April 19th at 5pm**

*Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.*

*At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).*

**A**

1. Let $R$ be the ring of all functions $f\colon \mathbb{R} \to \mathbb{R}$, with operations of pointwise addition and multiplication. Which of the following subsets are ideals?
   (a) $I_1 = \{f \in R\colon f(1) = 0\}$.
   (b) $I_2 = \{f \in R\colon f(1) = 0$ or $f(2) = 0\}$.
   (c) $I_3 = \{f \in R\colon f(1) = f(2)\}$.
2. Which of the following functions are ring homomorphisms?
   (a) $\phi\colon \mathbb{R} \to \mathbb{R}$ with $\phi(x) = |x|$.
   (b) $\phi\colon \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ with $\phi([x]) = [2x]$.
   (c) $\phi\colon \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ with $\phi([x]) = [4x]$.
   (d) $\phi\colon \mathbb{R}[x] \to \mathbb{R}[x]$ with $\phi(f) = \frac{\mathrm{d}f}{\mathrm{d}x}$.
3. Let $\mathbb{Z}[i] \subseteq \mathbb{C}$ be the subring consisting of all complex numbers $a + bi$ where $a, b \in \mathbb{Z}$. We call $\mathbb{Z}[i]$ the ring of *Gaussian integers*.
   (a) Let $I$ be the principal ideal generated by $2+2i$. How many elements are there in the quotient ring $\mathbb{Z}[i]/I$?
   (b) Is $I$ a prime ideal? Why or why not?

**B**

4. Let $F$ be a field, and let $R$ be a non-trivial ring. Prove that any surjective homomorphism $\phi\colon F \to R$ is an isomorphism.
5. Describe the set of maximal ideals in the ring $\mathbb{Z}/n\mathbb{Z}$ for all values $n \geq 0$.
6. An integral domain $R$ is called a *principal ideal domain* (PID) if every ideal $I \subseteq R$ with $I \neq R$ is principal, i.e. of the form $I = (a)$ for some $a \in R$. Prove that if $R$ is a PID then every non-zero prime ideal is maximal.
7. $(*)$ *Optional*: Let $R = \mathbb{R}[x, y]$, the ring of all real polynomials in two variables $x, y$, so a generic element of $R$ looks like

$$\sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} x^i y^j$$

for some non-negative integers $n$ and $m$, where the $a_{ij}$ are real numbers.

(a) Prove that $R/(x-y) \cong \mathbb{R}[x]$.

(b) What can you say about the quotient ring $R/(x^2-y)$?

(c) What can you say about the quotient ring $R/(x^2-y^2)$?