Math 350 – Groups, Rings and Fields – Homework 9 (§19)

Due: Friday April 26th at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. Write each of the following polynomials as a product of irreducible polynomials over the given field.
   (a) $2x^3 + x^2 + 2$ over $\mathbb{Z}/3$.
   (b) $x^3 + 3x^2 + x + 4$ over $\mathbb{Z}/5$.
   (c) $x^2 + 5$ over $\mathbb{Z}/7$.
   (d) $x^4 + x^3 + 2x^2 + x + 2$ over $\mathbb{Z}/3$.
   (e) $x^5 + x^2 - x - 1$ over $\mathbb{Z}/2$.
2. If R is a field, and $c \in R$ is a non-zero element, show that for any $f \in R[x]$ the elements f and cf generate the same principal idea in $R[x]$.
3. Let R and S be rings, and let $\phi\colon R \to S$ be a ring homomorphism. Show that the function $\phi_*\colon R[x] \to S[x]$ defined by
$$\phi_*(a_0 + a_1x + \cdots + a_nx^n) = \phi(a_0) + \phi(a_1)x + \cdots + \phi(a_n)x^n$$
is a ring homomorphism.

B

4. Let F be a field of characteristic zero (as in homework 8, problem 5). Prove that there exists a subfield of F isomorphic to $\mathbb{Q}$.
5. Let R be a field, let $f = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ and let a be a root of f. Define the *degree* of the root a to be the largest power m so that
$$f(x) = q(x)(x - a)^m.$$
   (a) Prove that if $f(x) = q(x)(x - a)^k$ and a is not a root of q then k is equal to the degree of a.
   (b) Show that the degree of a is greater than 1 if and only if a is a root of the derivative
$$f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$
6. Let F be a finite field with q elements.
   (a) Show that $a^{q-1} = 1$ for all $a \neq 0$ in F.
   (b) If $f \in F[x]$ show that there exists another polynomial $\widetilde{f} \in F[x]$ with either $\widetilde{f} = 0$ or $\deg(\widetilde{f}) < q$ so that $f(a) = \widetilde{f}(a)$ for all $a \in F$.

(c) Show that if $f, g \in F[x]$ with $\deg(f)$ and $\deg(g) < q$ satisfy $f(a) = g(a)$ for all $a \in F$ then $f = g$.

7. $(*)$ *Optional*: Let R be a field and let $R[x, y] = R[x][y]$ denote the ring of polynomials in two variables with coefficients in R. Prove that $f(x, y) = x^2 + y^2 - 1$ is irreducible unless the characteristic of R equals two. What happens in that case?