# Math 405 – Lie Groups and Lie Algebras – Homework 1

**Due: Friday February 7th**

*Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.*

*At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).*

1. Give an example of a subgroup $G \subseteq GL(n, \mathbb{R})$ for some $n \in \mathbb{N}$ that is *not* closed.

2. Let $V$ be a real vector space of dimension $n$. Prove that the groups $\mathrm{GL}(V)$ and $\mathrm{GL}(n, \mathbb{R})$ are isomorphic. You may use facts about linear maps and matrices from linear algebra without proof as long as you provide a clear statement.

3. Complete the argument from class that $\mathrm{GL}(n, \mathbb{C})$ is a matrix Lie group for all $n \in \mathbb{N}$.

4. Prove that $\mathbb{R}^n$ is a matrix Lie group. (Hint: you can find an injective group homomorphism $\phi \colon \mathbb{R}^n \to \mathrm{GL}(n+1, \mathbb{R})$ whose image is closed. Try the case $n = 1$ first).

5. (a) A *permutation matrix* in $\mathrm{GL}(n, \mathbb{R})$ is a matrix where each row and column contains a single 1 and all other entries 0. In other words, it is a matrix obtained from the identity matrix by permuting the rows or permuting the columns. Prove that the set of permutation matrices is a subgroup isomorphic to the symmetric group $S_n$. Deduce that $S_n$ is (isomorphic to) a matrix Lie group.

   (b) *Cayley's theorem* says that every finite group $G$ with $|G| = N$ is isomorphic to a subgroup of the symmetric group $S_N$. Use Cayley's theorem to prove that every finite group is isomorphic to a matrix Lie group.

6. $(*)$ *Optional.* We say a matrix Lie group $G$ is *connected* if there is no proper non-empty subset $U \subseteq G$ that is both open and closed. Prove that every connected Lie group is generated (as an ordinary group) by the neighborhood $N_\varepsilon(I) \cap G$ of the identity for any $\varepsilon > 0$.