Math 405 – Lie Groups and Lie Algebras – Homework 10

Due: Friday April 25th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra. Define the *normalizer* of $\mathfrak{h}$ to be

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \colon [X, Y] \in \mathfrak{h} \text{ for all } Y \in \mathfrak{h}\}.$$

   (a) Prove that $N_{\mathfrak{g}}(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$, and that if $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{k}$ is an ideal then $\mathfrak{k} \subseteq N_{\mathfrak{g}}(\mathfrak{h})$.

   (b) If $\mathfrak{g}$ is semisimple and $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, prove that $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$.

2. Prove that $X \in \mathfrak{sl}(n, \mathbb{C})$ is a regular element if and only if all the eigenvalues of X are distinct.

3. Let $\mathfrak{g}$ be a semisimple Lie algebra. The Killing form defines an inner product on $\mathfrak{h}^*$ using the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ sending α to H_α. In other words we can define

$$\kappa(\alpha, \beta) = \kappa(H_\alpha, H_\beta).$$

   (a) For roots α, β of $\mathfrak{g}$ prove that

$$\frac{\kappa(\alpha, \beta)}{\kappa(\alpha, \alpha)} \in \frac{1}{2}\mathbb{Z}.$$

   (b) If $\alpha \in \Phi$, define a reflection operator $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\beta) = \beta - \frac{2\kappa(\alpha, \beta)}{\kappa(\alpha, \alpha)}\alpha.$$

   Prove that $s_\alpha(\beta) \in \Phi$.

   (c) The elements s_α define a subgroup of $\mathrm{O}(r, \mathbb{R})$ where $r = \mathrm{rank}(\mathfrak{g})$ called the *Weyl group* of $\mathfrak{g}$. Determine the Weyl groups of $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$.

4. Let $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$.

   (a) Find a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (you may assume that $\mathrm{rank}(\mathfrak{g}) = 2$).

   (b) Determine the set $\Phi \subseteq \mathfrak{h}^*$ of roots of $\mathfrak{g}$. For each root $\alpha \in \Phi$ find a generator for the root space $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$.

   (c) Determine the Weyl group of $\mathfrak{g}$.

   (d) $(*)$ *Optional*: Let $\mathfrak{g}^\vee = \mathfrak{sp}(4, \mathbb{C})$, let $\mathfrak{h}^\vee$ be a Cartan subalgebra, and let $\Phi^\vee \subseteq (\mathfrak{h}^\vee)^*$ be the set of roots. Find an explicit isomorphism between $(\mathfrak{h})^*$ and $(\mathfrak{h}^\vee)^*$ sending Φ to $\Phi^\vee$.