Math 405 – Lie Groups and Lie Algebras – Homework 2

Due: Friday February 14th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Construct an isomorphism of groups from $\mathrm{U}(1)$ to $\mathrm{SO}(2, \mathbb{R})$.

2. (a) If p, q are any quaternions prove that
$$\overline{p \cdot q} = \overline{q} \cdot \overline{p}.$$
   (b) Let q be a quaternion. Let L_q denote the $\mathbb{R}$-linear map $\mathbb{H} \to \mathbb{H}$ defined by $L_q(x) = qx$. Let R_q denote the $\mathbb{R}$-linear map $\mathbb{H} \to \mathbb{H}$ defined by $R_q(x) = xq$. Check that $\det(L_q) = \det(R_q) = \|q\|^4$.

3. (a) Let G_1, G_2 be matrix Lie groups. Prove that the product $G_1 \times G_2$ is also a matrix Lie group.
   (b) Construct a homomorphism $\pi\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{O}(4, \mathbb{R})$ that is not the trivial homomorphism when restricted to either $\mathrm{SU}(2)$ factor (*Hint*: use the maps L_q, R_q from problem 1b).
   (c) Describe $\ker(\pi)$.
   (d) Use the results from Stillwell §2.5 to describe $\mathrm{Image}(\pi)$.

4. Let $\mathrm{Mat}_n(\mathbb{H})$ denote the set of $n \times n$ matrices with elements in $\mathbb{H}$.
   (a) Recall the injective map $\phi\colon \mathbb{H} \to \mathrm{Mat}_2(\mathbb{C})$ that we defined in class. By applying ϕ to each element of $A \in \mathrm{Mat}_n(\mathbb{H})$ define an injective ring homomorphism
$$i\colon \mathrm{Mat}_n(\mathbb{H}) \to \mathrm{Mat}_{2n}(\mathbb{C}),$$
   and check that the image of i is closed.
   (b) Define the *determinant* of $A \in \mathrm{Mat}_n(\mathbb{H})$ to be the determinant of the complex matrix $i(A)$. Use this to define $\mathrm{GL}(n, \mathbb{H})$ and $\mathrm{SL}(n, \mathbb{H})$ and prove that they are matrix Lie groups.
   (c) Suppose you tried to define a determinant function on $\mathrm{Mat}_2(\mathbb{H})$ by the usual formula
$$\det_{\mathbb{H}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$
   and then tried to define $\mathrm{GL}(2, \mathbb{H}) = \{A \in \mathrm{Mat}_2(\mathbb{H})\colon \det_{\mathbb{H}}(A) \neq 0\}$. Why is this not the right thing to do?

5. $(*)$ *Optional:* Construct a homomorphism $\mathrm{SU}(4) \to \mathrm{O}(6, \mathbb{R})$ with kernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (*Hint:* Use the conjugation action on the set $W_{\mathbb{C}}$ of 4×4 complex matrices such that $A = -A^T$. You'll need to find a suitable 6 *real*-dimensional subspace of $W_{\mathbb{C}}$ that this action preserves.)

   In fact (although you don't need to prove this) the image of this homomorphism is exactly $\mathrm{SO}(6, \mathbb{R})$.