Math 405 – Lie Groups and Lie Algebras – Homework 3

Due: Friday February 21st

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Recall that $\tau_r \colon \mathbb{H} \to \mathbb{H}$ is defined by $\tau_r(a + ib + jc + kd) = a + ib - jc + kd$.
   (a) Prove that $\tau_r(pq) = \tau_r(q)\tau_r(p)$ for $p, q \in \mathbb{H}$.
   (b) Prove that
   $$\mathrm{O}(n, \mathbb{H}) = \{A \in \mathrm{GL}(n, \mathbb{H}) \colon \tau_r(A)^T A = I\}$$
   is a subgroup of $\mathrm{GL}(n, \mathbb{H})$.
2. Prove that $A \in \mathrm{O}(n, \mathbb{R})$ if and only if the rows of A form an orthonormal basis for $\mathbb{R}^n$, if and only if the columns of A form an orthonormal basis for $\mathbb{R}^n$.
3. A matrix Lie group $G \subseteq \mathrm{GL}(n, \mathbb{R})$ is called *compact* if its matrix entries are *bounded*. In other words, there exists $R \in \mathbb{R}_{>0}$ so that for all $A \in G$ and all $1 \leq i, j \leq n$ we have
   $$|A_{ij}| \leq R.$$
   (a) Prove that $\mathrm{O}(n, \mathbb{R})$ is compact for all $n \in \mathbb{N}$.
   (b) Prove that $\mathrm{O}(n, \mathbb{C})$ is *not* compact for $n > 1$.
   (c) Prove that $\mathrm{O}(p, q)$ is *not* compact if p and q are both ≥ 1 (*Hint*: it's enough to check that $\mathrm{O}(1, 1)$ is not compact).
4. In this exercise we will prove the *linear Darboux theorem*. Let $V = k^n$ where $k = \mathbb{R}$ or $\mathbb{C}$, and let ω be a bilinear form on V that is *skew-symmetric*: $\omega(v, w) = -\omega(w, v)$.
   (a) A subspace $W \subseteq V$ is called *isotropic* if $\omega(v, w) = 0$ for all $v, w \in W$. Prove that there exists an isotropic subspace of dimension d with $2d \geq n$.
   (b) Let W be a maximal isotropic subspace (an isotropic subspace whose dimension is as large as possible). Let $p_1, \ldots, p_d$ be a basis for W. Show that you can extend this to a basis $p_1, \ldots, p_d, q_1, \ldots, q_m$ for V with the following two properties:
   - $\mathrm{Span}(q_1, \ldots, q_m) \subseteq V$ is isotropic.
   - $\omega(p_i, q_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$

   (c) Deduce that ω is equivalent (under change of basis) to
   $$\omega_{2m}(v, w) = v^T J_{2m} w$$
   for $2m \leq n$, where
   $$J_{2m} = \begin{pmatrix} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. $(*)$ *Optional:* Prove that
$$\mathrm{O}^*(2n, \mathbb{C}) \cong \mathrm{U}(n,n) \cap \mathrm{O}(2n, \mathbb{C})$$
where $\mathrm{O}^*(2n, \mathbb{C}) = \mathrm{O}(n, \mathbb{H})$ is the subgroup of $\mathrm{GL}(n, \mathbb{H})$ preserving the standard bilinear form on $\mathbb{H}^n$.