Math 405 – Lie Groups and Lie Algebras – Homework 4

Due: Friday February 28th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Compute the dimension of the following Lie algebras as real vector spaces.
   (a) $\mathfrak{so}(n, \mathbb{R})$.
   (b) $\mathfrak{su}(n)$.
   (c) $\mathfrak{sp}(2n, \mathbb{R})$.
   (d) $\mathfrak{sl}(n, \mathbb{H})$.
2. We've been studying $\mathfrak{g} = T_I G$, but in fact the tangent spaces at *all* points $g \in G$ are isomorphic. So fix an element $g \in G$ and prove the following.
   (a) Use left multiplication to define a bijection between the set of paths in G at g and the set of paths at $I \in G$.
   (b) Use your bijection to define a linear isomorphism $\lambda_g \colon \mathfrak{g} \to T_g G$.
   (c) You could've used the right multiplication map instead to define an alternative isomorphism $\rho_g \colon \mathfrak{g} \to T_g G$. Give an example in which $\lambda_g \neq \rho_g$.
3. If G is a matrix Lie group with Lie algebra $\mathfrak{g}$, define the *complexification* of $\mathfrak{g}$ to be the complex vector space
$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}.$$
   So if $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$ then $\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{gl}(n, \mathbb{C})$
   (a) Prove that the complexification of $\mathfrak{sl}(n, \mathbb{R})$ is $\mathfrak{sl}(n, \mathbb{C})$.
   (b) Prove that the complexification of $\mathfrak{su}(n)$ is also $\mathfrak{sl}(n, \mathbb{C})$ (if you get stuck look at Stillwell page 110).
   (c) Consider $\mathfrak{sl}(n, \mathbb{C}) \subseteq \mathfrak{gl}(2n, \mathbb{R})$. What is the complexification of $\mathfrak{sl}(n, \mathbb{C})$ as a subalgebra of $\mathfrak{gl}(2n, \mathbb{C})$?
4. Define a bilinear form κ on $\mathfrak{su}(2)$ by
$$\kappa(X, Y) = \frac{1}{2}\mathrm{Tr}(XY^\dagger).$$
   Prove that κ is symmetric, nondegenerate and positive definite, and that it is invariant under conjugation by $g \in SU(2)$, i.e.
$$\kappa(gXg^{-1}, gYg^{-1}) = \kappa(X, Y).$$
5. Let $\phi \colon G \to H$ be a differentiable homomorphism of matrix Lie groups. We can define a map $\phi_* \colon \mathfrak{g} \to \mathfrak{h}$ between their Lie algebras as follows: if $X \in \mathfrak{g}$, let $P_X : \mathbb{R} \to G$ be the associated one-parameter subgroup. We define $\phi_*(X)$ by the identity
$$\phi \circ P_X = P_{\phi_*(X)}.$$

(a) If $\phi\colon G \to H$ and $\psi\colon H \to K$ are homomorphisms, prove that $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

(b) Prove that ϕ_* is a linear map. You may assume that

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) \exp(tY) = \frac{\mathrm{d}}{\mathrm{d}t} \exp(t(X+Y)) \text{ when } t = 0.$$

(c) Let $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$ be the surjective homomorphism we constructed in class. Prove that $\pi_*\colon \mathfrak{su}(2) \to \mathfrak{so}(3, \mathbb{R})$ is an isomorphism.