Math 405 – Lie Groups and Lie Algebras – Homework 5

Due: Friday March 7th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. (a) Define the *center* $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to be

$$Z(\mathfrak{g}) = \{Z \in \mathfrak{g} \colon [Z, X] = 0 \,\forall X \in \mathfrak{g}\}.$$

   Prove that the center of a Lie algebra is an ideal.

   (b) Let $\mathfrak{z} \subseteq \mathfrak{gl}(n, \mathbb{R})$ be the subalgebra of scalar multiples of the identity matrix. Prove that

$$Z(\mathfrak{gl}(n, \mathbb{R})) = \mathfrak{z}.$$

   (c) Define the *direct sum* of two Lie algebra $\mathfrak{g}, \mathfrak{h}$ to be the vector space $\mathfrak{g} \oplus \mathfrak{h}$ with Lie bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]).$$

   Prove that $\mathfrak{gl}(n, \mathbb{R}) \cong \mathfrak{z} \oplus \mathfrak{sl}(n, \mathbb{R})$.

2. (a) Prove that vector cross product on $\mathbb{R}^3$ satisfies the axioms of a Lie algebra.

   (b) Prove that $\mathbb{R}^3$ with Lie bracket given by the cross product is isomorphic to $\mathfrak{so}(3, \mathbb{R})$.

   (c) Prove that if $\phi \colon \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras and $\dim(\mathfrak{g}) = \dim(\mathfrak{h}) = d$, then the images of the adjoint homomorphisms

$$\mathrm{ad}^{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{gl}(d, \mathbb{R}), \quad \mathrm{ad}^{\mathfrak{h}} \colon \mathfrak{h} \to \mathfrak{gl}(d, \mathbb{R})$$

   are conjugate:

$$\mathrm{Image}(\mathrm{ad}^{\mathfrak{h}}) = A \mathrm{Image}(\mathrm{ad}^{\mathfrak{g}}) A^{-1}$$

   for some $A \in \mathrm{GL}(d, \mathbb{R})$.

   (d) Hence prove that $\mathbb{R}^3$ with Lie bracket given by the cross product is *not* isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

3. Prove the first isomorphism theorem for Lie algebras.

4. Let $\mathfrak{g}$ be a Lie algebra and fix $X \in \mathfrak{g}$. Prove that the vector subspace spanned by the eigenvectors of ad_X is a Lie subalgebra of $\mathfrak{g}$.

5. $(*)$ *Optional*: Prove that $\mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C})$.