Math 405 – Lie Groups and Lie Algebras – Homework 6

Due: Friday March 14th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Prove that the Lie algebra $\mathfrak{n}_n$ of strictly upper triangular matrices is nilpotent for all n.

2. For a Lie algebra $\mathfrak{g}$, prove that $\mathrm{rad}(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ has no non-zero abelian ideals.

3. If $\mathfrak{n}$ is a nilpotent Lie algebra, prove that there exists an ideal $I \subseteq \mathfrak{n}$ of codimension 1 (that is, where the quotient $\mathfrak{n}/I$ has dimension 1). Describe such an ideal when $\mathfrak{n} = \mathfrak{n}_n$.

4. Prove that every 3-dimensional real Lie algebra $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{g}'$ is isomorphic to either $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{so}(3,\mathbb{R})$ (Hint: start with the analogous argument over $\mathbb{C}$ in Erdmann–Wildon section 3.2.4).

5. Fix positive integers $k \leq n$. Let $\mathfrak{p} \subseteq \mathfrak{gl}(n,\mathbb{C})$ be the Lie algebra consisting of block upper-triangular matrices

$$\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)$$

where $A \in \mathfrak{gl}(k,\mathbb{C}), C \in \mathfrak{gl}(n-k,\mathbb{C})$ and $B \in \mathrm{Mat}_{n-k,k}(\mathbb{C})$. Show that $\mathrm{rad}(\mathfrak{p})$ is spanned by those matrices where $A = \lambda I, C = \lambda I$ for some $\lambda \in \mathbb{C}$. You may use the fact that $\mathfrak{sl}(k,\mathbb{C})$ has no non-trivial ideals (we will prove this in class soon).

6. If $\mathfrak{g}$ is a Lie algebra its algebra $\mathrm{der}(\mathfrak{g})$ of *derivations* consists of all linear maps $\delta \colon \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)].$$

Its subalgebra of *inner* derivations $\mathrm{inn}(\mathfrak{g})$ consists of those linear maps of the form ad_X for all $X \in \mathfrak{g}$.

   (a) Prove that $\mathrm{der}(\mathfrak{g})$ is a Lie algebra and $\mathrm{inn}(\mathfrak{g}) \subseteq \mathrm{der}(\mathfrak{g})$ is an ideal for all Lie algebras $\mathfrak{g}$.

   (b) Let $\mathfrak{b}$ denote the non-abelian two-dimensional Lie algebra. Prove that $\mathrm{der}(\mathfrak{b}) = \mathrm{inn}(\mathfrak{b})$.

   (c) $(*)$ *Optional*: Let $\mathfrak{g} = \mathfrak{n}_3$ be the real Heisenberg algebra. Show that $\dim(\mathrm{der}(\mathfrak{g})) = 6, \dim(\mathrm{inn}(\mathfrak{g})) = 2$ and $\mathrm{der}(\mathfrak{g})/\mathrm{inn}(\mathfrak{g}) \cong \mathfrak{gl}(2,\mathbb{R})$.