Math 405 – Lie Groups and Lie Algebras – Homework 7

Due: Friday April 4th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Let $\mathfrak{b}$ be a solvable Lie algebra. Prove that there exists a sequence of ideals
$$0 = I_N \subseteq I_{N-1} \subseteq \cdots \subseteq I_1 \subseteq I_0 = \mathfrak{b}$$
where I_k/I_{k+1} is one-dimensional for all $k < N$.

2. Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and let $V = \mathbb{C}^{2n}$. Let ρ denote the representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ given by the inclusion. Prove that
$$B_V(X, Y) = \mathrm{Tr}(\rho(X)\rho(Y))$$
is non-degenerate.

3. Compute the Killing form of the following Lie algebras with respect to a basis of your choice.
   (a) The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.
   (b) The Heisenberg algebra $\mathfrak{n}_3 \subseteq \mathfrak{gl}(3, \mathbb{C})$.

4. Let V be a finite-dimensional vector space.
   (a) Show that if $X, Y \in \mathfrak{gl}(V)$ then
   $$XY^m = Y^m X + \sum_{k=1}^{m} \binom{m}{k} Y^{m-k} X_k$$
   where $X_1 = [X, Y]$ and $X_k = [X_{k-1}, Y]$ for $k > 1$.
   (b) Deduce that if $Y \in \mathfrak{gl}(V)$ then $\mathrm{ad}(Y^m)$ can be written as a linear combination of $Y^{m-k}\mathrm{ad}(Y)^k$ for $k = 1, \ldots, m$ [1].

5. (a) If $\mathfrak{g}$ is a complex semisimple Lie algebra prove that $\mathfrak{g} = \mathfrak{g}'$.
   (b) $(*)$ *Optional*: Is the converse true? That is, does there exist a finite-dimensional complex Lie algebra where $\mathfrak{g} = \mathfrak{g}'$ but $\mathfrak{g}$ is not semisimple?

6. A bilinear form B on a real vector space V is *negative semi-definite* if $B(v, v) \leq 0$ for all $v \in V$ and *negative definite* if $B(v, v) < 0$ for all $v \neq 0$.
   (a) Prove that the Killing form on $\mathfrak{so}(n, \mathbb{R})$ and $\mathfrak{su}(n)$ is negative definite for all $n \geq 1$. You may assume that:
   $$\kappa(X, Y) = \lambda \mathrm{Tr}(XY)$$
   for some $\lambda \in \mathbb{R}$, where the trace is taken over $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively.

[1] In class I claimed erroneously that $\mathrm{ad}(Y)^m = \mathrm{ad}(Y^m)$, this exercise gives the correct statement.

(b) Prove that the Killing form on $\mathfrak{u}(n)$ is negative semi-definite but not negative definite.

(c) $(*)$ *Optional*: Let $\mathfrak{g}$ be semisimple and suppose $\mathfrak{g} = \mathrm{Lie}(G)$. Consider the homomorphism $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. If κ has signature (p, q) prove that $\mathrm{Image}(\mathrm{Ad})$ is a closed subgroup of $\mathrm{O}(p, q)$ with $q > 0$. Hence deduce that if κ is negative definite then G is compact.