Math 405 – Lie Groups and Lie Algebras – Homework 8

Due: Friday April 11th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. A representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called *faithful* if it is injective as a homomorphism. Let $\mathfrak{b}$ be a non-abelian solvable complex Lie algebra. Prove that $\mathfrak{b}$ has no faithful finite-dimensional irreducible representations.

2. Prove the first isomorphism theorem for representations of a Lie algebra $\mathfrak{g}$. In other words, if $\phi\colon V \to W$ is a homomorphism of $\mathfrak{g}$-representations, prove that $\ker(\phi) \subseteq V$ and $\mathrm{Image}(\phi) \subseteq W$ are subrepresentations and $\mathrm{Image}(\phi) \cong V/\ker(\phi)$.

3. Let $\mathfrak{g}$ be a complex Lie algebra. Prove that $\mathfrak{g}$ has infinitely many non-isomorphic one-dimensional representations if and only if $\mathfrak{g}' \neq \mathfrak{g}$.

4. Determine the Casimir operator for the following representations.
   (a) The adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$.
   (b) The defining representation of $\mathfrak{so}(4, \mathbb{C})$.

5. Recall the definition of $\mathrm{der}(\mathfrak{g}), \mathrm{inn}(\mathfrak{g})$ from homework 6.
   (a) For a Lie algebra $\mathfrak{g}$ and $\delta \in \mathrm{der}(\mathfrak{g})$, show that the following formula defines a $\mathfrak{g}$-action on $\mathbb{R} \oplus \mathfrak{g}$:
   $$\alpha(X, (c, Y)) = (0, c\delta(X) + [X, Y]).$$
   (b) Check that the adjoint representation $\mathfrak{g}$ is a submodule of $\mathbb{R} \oplus \mathfrak{g}$.
   (c) Suppose $\mathfrak{g}$ is finite-dimensional and semisimple. Use Weyl's theorem and parts a and b to deduce that $\mathrm{inn}(\mathfrak{g}) = \mathrm{der}(\mathfrak{g})$.

6. $(*)$ *Optional*: Consider the inclusion $\mathfrak{sl}(2, \mathbb{C}) \subseteq \mathfrak{gl}(n, \mathbb{C})$ for $n \geq 2$ as block matrices in the upper left corner. Show that $\mathfrak{sl}(2, \mathbb{C})$ acts on $\mathfrak{gl}(n, \mathbb{C})$ by the adjoint action. Decompose $\mathfrak{gl}(n, \mathbb{C})$ into a direct sum of irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$.