Math 405 – Lie Groups and Lie Algebras – Homework 9

Due: Friday April 18th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. If V is a finite-dimensional representation of a semisimple Lie algebra $\mathfrak{g}$, prove that the Casimir operator C_V is independent of the choice of orthonormal basis.

2. Prove that the condition $[D, N] = 0$ in the Jordan decomposition is necessary for uniqueness. That is, find a matrix $X \in \mathfrak{gl}(n, \mathbb{C})$ and a decomposition $X = D + N$ where $D \in \mathfrak{gl}(n, \mathbb{C})$ is diagonalizable and $N \in \mathfrak{gl}(n, \mathbb{C})$ is nilpotent, but where $D + N$ is *not* the Jordan decomposition (and necessarily $[D, N] \neq 0$).

3. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be finite-dimensional complex Lie algebras, and let $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a surjective homomorphism. If $X \in \mathfrak{g}_1$ has abstract Jordan decomposition $X = D + N$, prove that $\phi(X)$ has abstract Jordan decomposition $\phi(D) + \phi(N)$.

4. Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$, not necessarily finite-dimensional, but suppose that there exists a non-zero h-eigenvector $v \in V$ with eigenvalue $\lambda \in \mathbb{C}$ such that $e \cdot v = 0$.

   (a) Show that
   $$ef^k \cdot v = k(\lambda - k + 1)f^{k-1} \cdot w.$$
   and
   $$e^k f^k \cdot v = (k!)^2 \binom{\lambda}{k} v$$
   where $k \in \mathbb{Z}_{>0}$ and $\binom{\lambda}{k} = \frac{1}{k!}\lambda(\lambda - 1) \cdots (\lambda - k + 1)$.

   (b) Deduce that V is finite-dimensional only if λ is a non-negative integer.

5. (a) Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Decompose the representation $\mathrm{Sym}^2(\mathfrak{g})$ into a direct sum of irreducible representations.

   (b) Decompose the representation $\mathrm{Sym}^2(V_4)$ into a direct sum of irreducible representations.

   (c) $(*)$ *Optional:* Decompose the representation $\mathrm{Sym}^n(\mathfrak{g})$ into a direct sum of irreducible representations for all n.