Math 455 – Algebraic Topology – Homework 1

Due: Friday February 10th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. (a) Show that a finite intersection of open subsets of $\mathbb{R}^n$ is still open.

   (b) If V is any subset of $\mathbb{R}^n$ of the form
   $$\{f(x_1, \ldots, x_n) = 0\},$$
   where f is a polynomial function $\mathbb{R}^n \to \mathbb{R}$, find an infinite sequence $U_1, U_2, U_3, \ldots$ of open subsets of $\mathbb{R}^n$ with the property that $\bigcap_{j=1}^{\infty} U_j = V$.

2. (*Furstenburg's proof of the infinitude of the primes.*) We define the *arithmetic sequence* topology τ on $\mathbb{Z}$ as follows. A subset $U \subseteq \mathbb{Z}$ is in τ if U is either empty or a union of arithmetic sequences
   $$S(a, d) = \{\ldots, a - 3d, a - 2d, a - d, a, a + d, a + 2d, a + 3d, \ldots\}$$
   where a, d are integers and $d \neq 0$.

   (a) Prove that τ is a topology.

   (b) Prove that if $V \subseteq \mathbb{Z}$ is a cofinite subset (a set whose complement is finite) other than $\mathbb{Z}$ itself, then V is not closed.

   (c) Prove that
   $$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} S(0, p)$$
   where the union is taken over all prime numbers p, and hence deduce that there are infinitely many prime numbers.

3. Let β be the set of all subsets of $\mathbb{R}$ of the form $[a, b)$ for $a < b$.

   (a) Prove that β is a base for some topology τ.

   (b) Prove that in any such topology τ, every $U \in \beta$ is both open and closed.

   (c) Prove that τ has no countable base.

4. (*The Cantor Set.*) Let $I = [0, 1] \subseteq \mathbb{R}$, equipped with the subspace topology. We define a sequence of subsets of I by the following procedure.
   $$\begin{aligned}
   C_1 &= I \setminus (1/3, 2/3)\\
   C_2 &= C_1 \setminus ((1/9, 2/9) \cup (7/9, 8/9))\\
   C_3 &= C_2 \setminus ((1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20/27) \cup (25/27, 26/27))
   \end{aligned}$$

and so on. So each set C_n is a union of 2^n closed intervals, and to obtain C_{n+1} we remove the *open middle third* from each of those intervals. Define the *Cantor set* C to be the intersection $\bigcap_{n=1}^{\infty} C_n$.

(a) Prove that $C \subseteq I$ is a closed set.

(b) Prove that C is uncountable. Hint: you can describe C in an alternative way by thinking about real numbers written in base 3.

(c) Prove that C does not contain any non-empty open interval (a, b).