Math 455 – Topology – Homework 10

Due: Friday May 5th

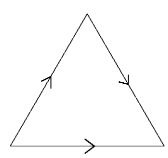
Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

Note: problems marked with a $(*)$ are *optional*, and will earn you extra credit. Problems marked with a $(\dagger)$ are graded on *completion only*, meaning that if you make a serious attempt at completing all the parts you will earn full credit.

1. Prove that finite covering maps (finite meaning that the discrete space D is finite) satisfy the "two-out-of-three property". In other words, given continuous functions $f\colon X \to Y$ and $g\colon Y \to Z$. If any two members of the set $\{f, g, g \circ f\}$ are finite covering maps then so is the third.

2. The *topologist's dunce hat* is the topological space X obtained by taking a solid closed triangle in $\mathbb{R}^2$ and gluing all three of its sides together according the following picture:

   (a) Write the dunce hat X as a glued space of the form $S^1 \cup_{S^1} D^2$, where D^2 is the unit disk, using the inclusion $i\colon S^1 \to D^2$ of the boundary, and a continuous map $f\colon S^1 \to S^1$ that you must determine.

   (b) Show that the dunce hat is contractible (Hint: show that the function f that you wrote down in the first part is homotopic to the identity map.)

3. Let G be a finite group (viewed as a topological group with the discrete topology) acting on a Hausdorff topological space X.

   (a) Suppose that the action of G on X is *free*, meaning that for any $x \in X$, $g \cdot x = x$ only if $g = e$. Prove that for any $x \in X$, there exists an open neighborhood U_x so that $(g \cdot U_x) \cap (h \cdot U_x) = \varnothing$ for all $g \neq h$.

   (b) Under this assumption, prove that the identification map $p\colon X \to X/G$ is a covering map.

   (c) Deduce that there is a covering map from S^n to $\mathbb{RP}^n$ for any natural number n.

4. (a) Let $X = S^2/\sim$ where $\sim$ is the equivalence relation identifying the north and south poles. Show that X is homotopy equivalent to $S^2 \vee S^1$ (drawing pictures is sufficient).

   (b) Hence prove $\pi_1(X, x) \cong \mathbb{Z}$ for any point $x \in X$ (Hint: write X as a quotient of a simply connected space).

5. (†) *Completion grade only:* Draw pictures of spaces X that admit covering maps down to $S^1 \vee S^1$ until you get tired (at least five different ones). In your pictures, use two colors (or e.g. solid and dashed lines) to distinguish points lying in the pre-images of the two copies of S^1.