Math 455 – Algebraic Topology – Homework 2

Due: Friday February 17th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Define the *interior* $\mathring{Y}$ of a subset Y of a topological space X to be the union of all open subsets $U \subseteq Y$ (so, the largest open subset of Y).
   (a) Prove that for any space X and any closed subset $Y \subseteq X$, we have $\overline{\mathring{Y}} \subseteq Y$. Find a topological space X and a closed subset $Y \subseteq X$ with the property that $\overline{\mathring{Y}} \neq Y$.
   (b) Prove that for any space X and any open subset $Z \subseteq X$, we have $Z \subseteq \mathring{\overline{Z}}$. Find a topological space X and an open subset $Z \subseteq X$ with the property that $\mathring{\overline{Z}} \neq Z$.
2. (a) Prove that there is no homeomorphism $f\colon \mathbb{R} \to \mathbb{R} \sqcup \mathbb{R}$, where $\mathbb{R} \sqcup \mathbb{R}$ is the disjoint union of two copies of the real line (if you like you can identify $\mathbb{R} \sqcup \mathbb{R}$ with the subspace of $\mathbb{R}^2$ consisting of points (x, y) where either $y = 0$ or $y = 1$.)
   (b) Prove that there is no homeomorphism $f\colon \mathbb{R} \to X$, where X is the union of the x and y axes in $\mathbb{R}^2$, equipped with the subspace topology.
3. Do there exist two topological spaces X, Y that are not homeomorphic, but with the property that X is homeomorphic to a subspace of Y, and Y is homeomorphic to a subspace of X?
4. (Stereographic Projection.) Let $n \geq 1$. Recall the definition of the stereographic projection map

   $$\pi\colon S^n \setminus \{p\} \to \mathbb{R}^n - p,$$

   where $p = (0, \ldots, 0, 1)$. We define $\pi(x)$ to be the unique point where the line through x and p passes through the plane $\mathbb{R}^n - p$.
   (a) Prove that π is a homeomorphism.
   (b) In the case where $n = 2$, give an alternative description of the homeomorphism π and its inverse using formulas in Cartesian coordinates.
5. $(*)$ *Optional:* Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be subsets equipped with the subspace topology. Is it possible that X and Y are not homeomorphic to one another, but $X \times [0, 1] \subseteq \mathbb{R}^{n+1}$ and $Y \times [0, 1] \subseteq \mathbb{R}^{m+1}$ *are* homeomorphic to one another? A detailed proof is not required, but please give some geometric justification.