Math 455 – Topology – Homework 3

Due: Friday February 24th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. (a) Prove that if X is any Hausdorff topological space, all singleton subsets $\{x\} \subseteq X$ are closed.

   (b) Show that the converse to this statement is false by finding an example of a topological space in which all singleton subsets are closed that is not Hausdorff.

2. (a) Given two subsets A, B of a metric space X, define

$$d(A, B) = \inf_{a \in A, b \in B} d(a, b).$$

   Find two disjoint closed subsets A, B of $\mathbb{R}^2$ with the property that $d(A, B) = 0$.

   (b) Define the *diameter* of a subset A of a metric space to be

$$\operatorname{diam}(A) = \sup_{a, a' \in A} d(a, a').$$

   Prove that for any two disjoint closed subsets $A, B \subseteq \mathbb{R}^2$ such that $d(A, B) = 0$, both $\operatorname{diam}(A)$ and $\operatorname{diam}(B)$ are infinite.

3. In this problem you will prove the *Urysohn metrization theorem*, which says that every regular topological space with a countable base is metrizable (but not conversely).

   (a) Prove that every regular topological space with a countable base is normal.

   (b) Write $\mathbb{R}^\infty$ for the set of all sequences $(x_i)_{i \in \mathbb{N}}$ of real numbers. Define a metric on $\mathbb{R}^\infty$ by

$$d(x, y) = \sup_{n \in \mathbb{N}} \left(\frac{1}{n} \left(\min(1, |y_n - x_n|) \right) \right).$$

   Prove that d is indeed a metric.

   (c) If X is regular and admits a countable basis, prove that there exists a sequence $f_n \colon X \to [0, 1]$ of continuous functions such that for all $x \in X$ and any open neighborhood U of x, there is some n for which $f_n(x) > 0$ but $f_n(y) = 0$ for all y not in U. You may use the fact that Urysohn's lemma applies to all normal topological spaces.

   (d) Using the previous part, define a function $F \colon X \to \mathbb{R}^\infty$ by

$$F(x) = (f_1(x), f_2(x), \ldots).$$

   Prove that F is continuous and injective, and that $F(U)$ is open for all open subset $U \subseteq X$. Hence deduce Urysohn's metrization lemma.

4. (a) (Double origin topology). Let $X = \mathbb{R}^2 \cup \{0'\}$ with the following topology. Open subsets of $\mathbb{R}^2 \setminus \{0\}$ are the same as in the standard topology. Subsets containing 0 are open if they contain a subset of the form

$$U_\varepsilon = \{(x, y) \colon x^2 + y^2 < \varepsilon, y > 0\} \cup \{0\}$$

for some $\varepsilon > 0$ and subsets containing $0'$ are open if they contain a subset of the form

$$U'_\varepsilon = \{(x, y) \colon x^2 + y^2 < \varepsilon, y < 0\} \cup \{0'\}$$

for some $\varepsilon > 0$. Show that X is a Hausdorff space, and admits a countable basis, but is not metrizable.

(b) Give an example of a metrizable space that does not have a countable base.