Math 455 – Topology – Homework 4

Due: Friday March 10th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. We say a metric space (X, d) has the *Heine–Borel property* if every closed subset $V \subseteq X$ with $\operatorname{diam}(V) < \infty$ is compact. We proved that $\mathbb{R}^n$ with the standard metric does have the Heine–Borel property.
   (a) Let X be the space of continuous functions $[0,1] \to \mathbb{R}$, equipped with the metric
   $$d_\infty(f, g) = \sup_{x \in [0,1]} |g(x) - f(x)|.$$
   Show that (X, d_∞) does not have the Heine–Borel property.
   (b) Consider $\mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology. Show that $\mathbb{Q}$ with the standard metric $d(q_1, q_2) = |q_2 - q_1|$ does not have the Heine–Borel property.
2. We say a space X is *locally connected* if for all $x \in X$ and all open neighborhoods $x \in U$, we can find a connected open subneighborhood $x \in V \subseteq U$.
   (a) Show that the subset $X = 0 \cup \{1/n \colon n \in \mathbb{N}\} \subseteq \mathbb{R}$ is not locally connected with respect to the subspace topology.
   (b) Show that X is locally connected if and only if for all open subsets $U \subseteq X$, all connected components C of U are open.
3. (One-point Compactification). Let X be any topological space. Define the *one-point compactification* X^* of X as follows. The underlying set is $X \cup \{*\}$. A subset $U \subseteq X^*$ is open if either it is open in X, or it it has the form $\{*\} \cup V$ where $V \subseteq X$ is a set whose complement is closed and compact.
   (a) Prove that this satisfies the axioms of a topological space.
   (b) Use stereographic projection to prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.
4. (Stone-Čech Compactification). Let X be any normal topological space. Define the *Stone-Čech compactification* βX of X as follows.
   (a) Let $C(X)$ denote the set of all continuous functions $f \colon X \to [0,1]$. Denote by $[0,1]^{C(X)}$ the set of all functions $C(X) \to [0,1]$. Explain how to view this set as a product of copies of $[0,1]$, and hence equip it with the product topology. Deduce that $[0,1]^{C(X)}$ is a compact Hausdorff topological space.
   (b) We define a function $e \colon X \to [0,1]^{C(X)}$ by sending x to the function sending $f \in C(X)$ to $f(x)$ in $[0,1]$. Prove that e is continuous.

(c) Define βX as the closure of the image of e. Show that βX is a compact space, and that there is an injective continuous map $i\colon X \to \beta X$ (Note: like on the previous homework you may use Urysohn's lemma for normal topological spaces).

(d) (*Optional*): Prove that βX is the *universal* compactification of X. In other words, given any compact Hausdorff space K, and an injective continuous map $j\colon X \to K$, you can find a continuous map $g\colon K \to \beta X$ so that $g \circ j = i$.