Math 455 – Topology – Homework 5

Due: Friday March 24th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Recall that the *suspension* ΣX of a topological space X is the quotient $\Sigma X = X \times [-1,1]/\sim$, where $\sim$ is the equivalence relation defined by $(x,-1) \sim (x',-1)$ for all $x, x' \in X$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$
   (a) Prove that ΣS^{n-1} is homeomorphic to S^n, for all $n \geq 1$.
   (b) Prove that any quotient of a compact space is compact, and hence deduce that if X is compact then so is ΣX.
   (c) Prove that for all spaces X, the space ΣX is path-connected.

2. We say a space X is *locally path connected* if for every point $x \in X$ and every open neighborhood U of x, there is an open path-connected subset $V \subseteq U$ containing x (similar to the definition of *locally connected* from the last homework).
   (a) Prove that if X is connected and locally path-connected, then X is path-connected.
   (b) Show that $\mathbb{Z}$ with the cofinite topology is connected and locally connected, but not path-connected or locally path-connected.

3. (a) Let $X = \mathbb{R}^2$ with the standard topology. Define an equivalence relation on X by $(x,y) \sim (x',y')$ if $x - x'$ and $y - y'$ are both integers. Prove that $X/\sim$ is homeomorphic to the torus $T^2 = S^1 \times S^1$.
   (b) Let q be a fixed irrational number. Define a new equivalence relation $\sim'$ on $X/\sim$ by $[(x,y)] \sim' [(x',y')$ lie on a single line through $\mathbb{R}^2$ with slope q, for *some* representatives of the equivalence classes $[(x,y)]$ and $[(x',y')]$. Prove that $(X/\sim)/\sim'$ is not Hausdorff.

4. Define an *arc* from x to y in a space X to be a continuous injective map $\gamma\colon [0,1] \to X$ so that $\gamma(0) = x$ and $\gamma(y) = 1$ (this is different from the definition of a path, because of the additional assumption that γ is injective). We say X is *arc-connected* if every two points in X can be connected by an arc.
   (a) Prove that the line with two origins is path-connected but not arc-connected.
   (b) Hence find a continuous surjective function $f\colon Y \to X$ from an arc-connected space to a non-arc-connected space (Hint: show that the line with two origins is a quotient of an arc-connected space).
   (c) Show that if X is a connected open subspace of $\mathbb{R}^n$ then X is arc-connected.
   (d) $(*)$ *Optional:* Show that if X is a path-connected locally Euclidean subspace of $\mathbb{R}^n$ then X is arc-connected (in fact it turns out that every path-connected Hausdorff space is arc-connected, but it's tough to prove this).