Math 455 – Topology – Homework 6

Due: Friday March 31st

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Recall that the *wedge sum* of spaces $(X_i)_{i \in I}$ associated to a choice of point $x_i \in X_i$ for each i, is the following quotient space of the union of all the X_i:

$$\bigvee_{i \in I} X_i = \coprod_{i \in I} X_i / \sim$$

   where $x_i \sim x_j$ for all $i, j \in I$.

   (a) Prove that the quotient space $\mathbb{R}/\sim$ where $n \sim m$ if n, m are both integers, and the wedge sum $\bigvee_{n=1}^{\infty} S^1$ of countably many circles, are homeomorphic to one another.

   (b) Let X be the space from part a), and let

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

   be the subspace of $\mathbb{R}^2$ given as the union of circles Y_n, where Y_n is the circle with radius $1/n$ and center $(1/n, 0)$. Prove that X and Y are not homeomorphic.

2. Prove that real projective space $\mathbb{RP}^n$ is a *manifold* (in other words, prove that it is Hausdorff, locally Euclidean, and has a countable base).

3. (a) Let G be a group equipped with a topology. Prove that G is a topological group if and only if the function $f \colon G \times G \to G$ defined by $f(g, h) = g * h^{-1}$ is continuous.

   (b) Prove that a topological group G is Hausdorff if and only if the single element subset $\{e\} \subseteq G$ is closed (Hint: show that a space X is Hausdorff if the diagonal subspace $\Delta \subseteq X \times X$ defined by $\Delta = \{(x, x) \subseteq X \times X\}$ is closed).

   (c) Give an example of a non Hausdorff topological group.

4. In this problem, let p be a prime, and let $X = \prod_{n \in \mathbb{N}} \mathbb{Z}/p^n$. We view $\mathbb{Z}/p^n$ as a topological group with the discrete topology

   (a) Prove that the discrete topology and the product topology on X are *not* homeomorphic.

   (b) Equip X with the product topology. Define a subspace $\mathbb{Z}_p \subseteq X$ by

$$(x_i)_{i \in \mathbb{N}} \in \mathbb{Z}_p \text{ if } x_i \equiv x_{i+1} \bmod p^i \text{ for all } i \in \mathbb{N}.$$

   Prove that $\mathbb{Z}_p$ is a topological group, where the group operation is given by addition mod p^n on the factor $\mathbb{Z}/p^n$.

   (c) Prove that $\mathbb{Z}_2$ is homeomorphic to the Cantor set (from Homework 1).