Math 455 – Topology – Homework 7

Due: Friday April 14th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

Note: problems marked with a $(*)$ are *optional*, and will earn you extra credit. Problems marked with a $(\dagger)$ are graded on *completion only*, meaning that if you make a serious attempt at completing all the parts you will earn full credit.

1. (a) Define an action of $\mathbb{Z}$ on the space $X = \mathbb{R} \times [0,1]$ such that $X/\mathbb{Z}$ is homeomorphic to the Möbius strip.
   (b) Define an action of $\mathbb{Z}/2$ on the torus $T^2 = S^1 \times S^1$ such that $T^2/(\mathbb{Z}/2)$ is homeomorphic to a cylinder.
2. Recall that the *cone* of a space Y is defined by
$$\mathrm{Cone}(Y) = Y \times [0,1]/\sim$$
where $(y,0) \sim (y',0)$ for all $y, y' \in Y$. Prove that for any X, any two functions $f, g\colon X \to \mathrm{Cone}(Y)$ are homotopic.
3. (a) Prove that the function $f\colon S^1 \to S^1$ given by the identity, and $g\colon S^1 \to S^1$ given by
$$g(x) = \begin{cases} x^3 & \theta \le 2\pi/3 \\ x^{-3} & 2\pi/3 \le \theta \le 4\pi/3 \\ x^3 & \theta \ge 4\pi/3 \end{cases}$$
(where $x = e^{i\theta}$ in complex notation) are homotopic relative to the point $1 \in S^1$.
   (b) Let $f\colon S^1 \to S^1$ be a function that is not homotopic to the identity. Prove that there is some point $x \in S^1$ so that $f(x) = -x$.
   (c) Show that the converse to the statement in part b is false.
4. Let G be a topological group, and let $e \in G$ be the identity element.
   (a) Define a group structure on the set $\pi_0(G, e)$.
   (b) Explain how to construct a subgroup $G' \subseteq G$ for which there is a bijection
$$\phi\colon G/G' \to \pi_0(G, e).$$
   (c) $(*)$ *Optional*: If you know what a normal subgroup, prove that your subgroup $G' \subseteq G$ from the previous part is actually normal, and that the homeomorphism ϕ is additionally an isomorphism of groups.

5. (†) *Completion grade only:*

   (a) Prove that there is a homeomorphism between the projective space $\mathbb{RP}^n$ and the quotient space $\mathrm{SO}(n+1)mrO(n)$, where we identify $\mathrm{O}(n)$ as a subgroup of $\mathrm{SO}(n+1)$ by sending a matrix $C \in \mathrm{O}(n)$ to the $(n+1)\times(n+1)$ matrix
$$\left(\begin{array}{c|c} \det(C)^{-1} & 0 \\ \hline 0 & C \end{array}\right).$$

   (b) Prove that there is a homeomorphism between the projective space $\mathbb{RP}^n$ and the quotient space $\mathrm{O}(n+1)/(\mathrm{O}(1)\times\mathrm{O}(n)$.