Math 455 – Topology – Homework 8

Due: Friday April 21st

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

Note: problems marked with a $(*)$ are *optional*, and will earn you extra credit. Problems marked with a $(\dagger)$ are graded on *completion only*, meaning that if you make a serious attempt at completing all the parts you will earn full credit.

1. (a) If X is any space equipped with the discrete topology and $x \in X$ is any point, prove that $\pi_1(X, x)$ is trivial.

   (b) If X is any space equipped with the indiscrete topology and $x \in X$ is any point, prove that $\pi_1(X, x)$ is trivial.

2. Let X be a space, and let $p_1, p_2, p_3, p_4 \in X$ be any points.

   (a) If γ is a path from p_1 to p_2 and γ^{-1} is the reverse path from p_2 to p_1 (so $\gamma^{-1}(t) = \gamma(1-t)$, prove that $\gamma * \gamma^{-1}$ is homotopic to c_{p_1} relative to $\{0, 1\} \subseteq [0, 1]$, where c_{p_1} is the constant path at the point p_1

   (b) If γ_1 is a path from p_1 to p_2, γ_2 is a path from p_2 to p_3 and γ_3 is a path from p_3 to p_4, prove that $(\gamma_3 * \gamma_2) * \gamma_1$ and $\gamma_3 * (\gamma_2 * \gamma_1)$ are homotopic relative to $\{0, 1\} \subseteq [0, 1]$.

3. Let G be a topological group. Prove that $\pi_1(G, e)$ is abelian, meaning that for any two loops γ, γ' we have $[\gamma] * [\gamma'] = [\gamma'] * [\gamma]$.
   Hint: prove that both $\gamma * \gamma'$ and $\gamma' * \gamma$ are homotopic to the loop defined by
   $$\gamma \cdot \gamma'(t) = m(\gamma(t), \gamma'(t))$$
   where m is the group operation in G. In order to show this it is useful to notice that for any four loops $\alpha, \beta, \gamma, \delta$, we have
   $$(\gamma * \alpha) \cdot (\beta * \delta) = (\gamma \cdot \beta) * (\alpha \cdot \delta).$$

4. Recall that $\pi_2(X, x)$ is the set of homotopy classes of maps $S^2 \to X$ sending $(1, 0, 0)$ to $x \in X$, modulo homotopy relative to $\{(1, 0, 0)\}$.

   (a) Explain how to view $\pi_2(X, x)$ alternatively as homotopy classes of maps $[0, 1] \times [0, 1]$ to X sending the boundary of the square to the point x.

   (b) Define a group structure on $\pi_2(X, x)$ by gluing together two maps from the closed square into X along one of their edges.

(c) $(*)$ *Optional*: Give an argument demonstrating that the group $\pi_2(X, x)$ is always abelian. Your argument does not need to be very formal, drawing pictures will suffice.

5. $(\dagger)$ *Completion grade only:* Let $A = \{(r, \theta) \in \mathbb{R}^2 \colon 1 \leq r \leq 2\}$ be a closed annulus, and let $x = (1, 0) \in A$.

(a) Show that the function $f \colon A \to A$ defined by

$$f(r, \theta) = (r, \theta + 2\pi(r - 1))$$

is homotopic to the identity function.

(b) Suppose $g \colon A \to A$ is homotopic to the identity relative to the two boundary circles, and let $\gamma(t) = (t + 1, 0)$ and let $\delta(t) = g(\gamma(t))$ be two paths in A. Prove that the element $[\gamma^{-1} * \delta] \in \pi_1(A, x)$ is equal to the identity element.

(c) Conclude that f is *not* homotopic to the identity function relative to the two boundary circles of A. Hint: let $\pi \colon A \to S^1$ be the projection onto the θ coordinate. Prove that $\pi_*([\gamma^{-1} * \delta])$ is non-trivial.