Math 455 – Topology – Homework 9

Due: Friday April 28th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

Note: problems marked with a $(*)$ are *optional*, and will earn you extra credit. Problems marked with a $(\dagger)$ are graded on *completion only*, meaning that if you make a serious attempt at completing all the parts you will earn full credit.

1. Write out a proof of the homotopy lifting property for S^1. That is, if $\gamma, \gamma' \colon [0,1] \to S^1$ are two loops in S^1 that are homotopic relative to $\{0,1\}$ via a homotopy F, prove there exists a unique lift
$$\widetilde{F} \colon [0,1] \times [0,1] \to \mathbb{R}$$
that is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\gamma}'$ relative to $\{0,1\}$ (you can refer to the sketch argument in the textbook on page 98).

2. Describe the induced homomorphism $f_* \colon \mathbb{Z} \to \mathbb{Z}$ between fundamental groups associated to each of the following continuous maps $f \colon S^1 \to S^1$.
   (a) $f(e^{i\theta}) = e^{in\theta}$ for some $n \in \mathbb{Z}$.
   (b) $f(e^{i\theta}) = e^{i(2\pi-\theta)}$.
   (c) $f(e^{i\theta}) = \begin{cases} e^{i\theta} & 0 \leq \theta \leq \pi \\ e^{i(2\pi-\theta)} & \pi \leq \theta \leq 2\pi. \end{cases}$

3. Let X be a contractible space, meaning that X is homotopy equivalent to a point. Prove the following.
   (a) X is path-connected.
   (b) Any two continuous functions $f, g \colon Y \to X$ are homotopic, for any space Y.
   (c) Any two continuous functions $f, g \colon X \to Y$ are homotopic if and only if the space Y is path-connected.

4. A homotopy equivalence $p \colon X \to A, i \colon A \to X$ is called a *deformation retraction* if i is injective and $p \circ i$ is equal to the identity. We say that A is a *deformation retract* of X.
   (a) Show that S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$.
   (b) Show that $S^1 \vee S^1$ is a deformation retract of the torus with a point removed $T^2 \setminus \{p\}$.

5. $(\dagger)$ *Completion grade only:* Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial of degree $n \geq 1$.

(a) Suppose first that $\|a_0\| + \|a_1\| + \cdots \|a_{n-1}\| < 1$. Show that there is a homotopy between the restriction of p to S^1, and the function $f\colon S^1 \to \mathbb{C} \setminus \{0\}$ given by $f(z) = z^n$.

(b) Prove that if $\|a_0\| + \|a_1\| + \cdots \|a_{n-1}\| < 1$ then the polynomial p has at least one root (i.e. there exists z_0 so that $p(z_0) = 0$.

(c) Now deduce the general case, where $a_0, \ldots, a_{n-1}$ can be any complex numbers (this is the fundamental theorem of algebra).