# Homology and Cohomology – Week 1 Exercises

Read up to Proposition 1.4.4 in Riehl's *Category Theory in Context* (including the preface). If you find something confusing, try to find the corresponding section in Leinster's *Basic Category Theory* Section 1 and compare them. Many but not all of the exercises below come from one of those two books.

Think about these exercises, and write up solutions to three of them.

1. (a) Show that if a morphism $f\colon X \to Y$ in a category $\mathcal{C}$ has a left inverse $g\colon Y \to X$ and a right inverse $h\colon Y \to X$ (so $gf = \mathrm{id}_X, fh = \mathrm{id}_Y$) then $g = h$.
   (b) Give an example, in a category of your choice, of a morphism with a left inverse but not a right inverse.
   (c) Show that, when a (two-sided) inverse of a morphism $f\colon X \to Y$ in a category $\mathcal{C}$ exists, it is necessarily unique.
2. (a) Show that every group $G$, viewed as a one object category, is isomorphic to its own opposite category.
   (b) Give an example of a monoid $M$ that is not isomorphic to its own opposite category.
3. (a) Unpack what footnote 19 on page 12 is telling you.
   (b) Read the definition of the skeleton of a category $\mathcal{C}$ (Definition 1.5.15) and show that the statement "every small category has a skeleton" is equivalent to the statement that every epimorphism in $\mathrm{Set}$ is split.
4. Solve Riehl's Exercise 1.3.ix.
5. Solve Leinster's Exercise 1.3.31.
6. Let $\mathcal{C}$ be a category. An automorphism of $\mathcal{C}$ (meaning an invertible functor $F\colon \mathcal{C} \to \mathcal{C}$) is *inner* if it is naturally isomorphic to the identity functor.
   (a) Justify the name "inner" (consider the case where $\mathcal{C}$ is a group. An automorphism of a group is called *inner* if it is given by conjugation by a group element).
   (b) Show that the inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms.
   (c) Show that if $\mathcal{C} = \mathrm{Set}$ then every automorphism is inner (Hint: think about what an automorphism does to single element sets).
7. $(*)$ (*Challenge Problem*): Expand the following outline to show that if $\mathrm{Top}_{\mathrm{fin}}$ is the category of finite topological spaces, then the index of the group of inner automorphisms of $\mathrm{Top}_{\mathrm{fin}}$ in the group of all automorphisms of $\mathcal{C}$ is equal to 2.
   - Define the *Sierpiński space* $S$ to be the finite topological space with underlying set $\{0, 1\}$ and open sets $\{\varnothing, \{1\}, S\}$. Show that there are exactly three continuous functions $S \to S$, and show that any space with this property is homeomorphic to $S$.
   - Let $\mathcal{C}$ be any full subcategory of $\mathrm{Top}$ containing $S$. Deduce that for every automorphism $F$ of $\mathcal{C}$ we have $S \cong F(S)$.
   - Let $U\colon \mathcal{C} \to \mathrm{Set}$ be the forgetful functor. Show that there is a natural isomorphism $\alpha\colon U \to UF$.

- Suppose that $\mathcal{C}$ contains a space $X$ where not every union of closed sets is closed. Use the continuous functions $f\colon X \to S$ and the naturality square

$$\begin{tikzcd} U(X) \arrow[r, "\alpha_X"] \arrow[d, "U(f)"'] & UF(X) \arrow[d, "UF(f)"] \\ U(S) \arrow[r, "\alpha_S"] & UF(S) \end{tikzcd}$$

to show that $\alpha_S$ has to be continuous.
- Use the same idea to deduce that if $\mathcal{C} = \mathrm{Top}_{\mathrm{fin}}$ then there are two natural isomorphism classes of automorphisms.