Homology and Cohomology – Week 11 Exercises

Read Hatcher *Algebraic Topology* §3.3 and tom Dieck *Algebraic Topology* §18.1–18.3 (you might also look at §15.1, 15.2, 15.5 for a reference on terminology from smooth geometry).

Think about these exercises, and write up solutions to three of them. If you like you can also submit exercise solutions from previous weeks.

1. Hatcher page 257 problems 3 and 5.
2. Prove that a surface S is non-orientable if and only if there is a homeomorphism between the Möbius strip M and an open subspace of S (give a direct proof, don't use the classification of surfaces).
3. Describe the nondegenerate pairing on $\mathrm{H}^2(S^2 \times S^2; \mathbb{C})$ and on $\mathrm{H}^2(\mathbb{CP}^2; \mathbb{C})$.
4. Prove the *Poincaré lemma*:
$$\mathrm{H}^k_c(\mathbb{R}^n; \mathbb{R}) = \begin{cases} \mathbb{R} \text{ if } k = n \\ 0 \text{ otherwise.} \end{cases}$$
Give a direct argument, then an argument using Poincaré duality.
5. Hatcher page 259 problem 24.
6. Hatcher page 259 problem 26.
7. Prove that if $g < h$, and Σ_g is the orientable surface of genus g, then any continuous map $f\colon \Sigma_g \to \Sigma_h$ has degree 0, i.e. induces the zero map on H^2 (Hint: Show that there exists a nonzero $\alpha \in \mathrm{H}^1(\Sigma_h)$ with $f^*\alpha = 0$, then use facts about the Poincaré duality pairing to show there exists $\beta \in \mathrm{H}^1(\Sigma_h)$ with $\alpha \smile \beta \neq 0$, then use naturality of the cup product to show $\deg(f) = 0$.)