Homology and Cohomology – Week 2 Exercises

Read Chapter 1.1–1.4 of *Introduction to Homological Algebra* by Weibel `https://people.math.rochester.edu/faculty/doug/otherpapers/weibel-hom.pdf`. When you learn about the snake lemma (Lemma 1.3.2) don't forget to watch the proof given in 1980 Hollywood romantic comedy *It's My Turn* `https://www.youtube.com/watch?v=etbcKWEKnvg`. Read about my favorite example of a cochain complex, the de Rham complex $\Omega^\bullet(\mathbb{R}^n)$ in Bott and Tu's *Differential Forms in Algebraic Topology* chapter 1 `https://www.math.auckland.ac.nz/~hekmati/Books/BottTu.pdf`.

Think about these exercises, and write up solutions to three of them.

1. Prove that homology $\mathrm{H}_\bullet$ defines a functor from chain complexes of R-modules to R-modules.
2. Solve Weibel's exercise 1.1.7 to compute the homology of the tetrahedron.
3. Solve Weibel's exercise 1.4.1 on split exact sequences.
4. Suppose (V, d) is a chain complex over a field k that is finite-dimensional (meaning finite-dimensional in each degree, and non-zero in only finitely many degrees. Define the *Euler characteristic* $\chi(V)$ to be
$$\chi(V) = \sum_{n\in\mathbb{Z}} (-1)^n \dim(V_n).$$
   (a) Prove that U, V, W are finite-dimensional graded vector spaces and there is an exact sequence
$$\cdots \to U_n \to V_n \to W_n \to U_{n-1} \to V_{n-1} \to \cdots$$
   then $\chi(U) + \chi(W) = \chi(V)$.
   (b) For any finite-dimensional chain complex (V, d) prove that $\chi(V) = \chi(\mathrm{H}_\bullet(V))$.
5. (a) Suppose (C, d) is a chain complex defined over a field k. Show that (C, d) is chain homotopy equivalent to $(\mathrm{H}_\bullet(C), 0)$.
   (b) Replace the field k by $\mathbb{Z}$ and give a counterexample to the corresponding statement.
6. Let $X = \mathbb{R}^2 \setminus (0,0)$, the punctured plane.
   (a) Prove that $\mathrm{H}^0_{\mathrm{dR}}(X) \cong \mathbb{R}$
   (b) Prove that $\mathrm{H}^0_{\mathrm{dR}}(X) \cong \mathbb{R}$ by giving a non-trivial linear map $\mathrm{Z}^0_{\mathrm{dR}}(X) \to \mathbb{R}$ by integration around a curve around the origin, and checking that the kernel is $\mathrm{B}^0_{\mathrm{dR}}(X)$ (think back to multivariate calculus).