Homology and Cohomology – Week 2 Exercises

If you didn't already, read about my favorite example of a cochain complex, the de Rham complex $\Omega^\bullet(\mathbb{R}^n)$ in Bott and Tu's *Differential Forms in Algebraic Topology* chapter 1 `https://www.math.auckland.ac.nz/~hekmati/Books/BottTu.pdf`.

Think about these exercises, and write up solutions to three of them.

1. (a) Prove that chain homotopy equivalence is an equivalence relation.
   (b) Prove that the set of chain homotopy equivalence classes of chain maps $C_\bullet \to D_\bullet$ forms an abelian group.
2. Let C be the chain complex with $C_1 = \mathbb{Z}$, $C_0 = \mathbb{Z}$ with differential given by multiplication by n. Let $D = \mathbb{Z}/n$ viewed as a chain complex concentrated in degree zero. Prove that C and D are quasi-isomorphic (meaning there is a chain map between them inducing an isomorphism on cohomology) but not chain homotopy equivalent.
3. Let $X = \mathbb{R}^2 \setminus (0,0)$, the punctured plane.
   (a) Prove that $\mathrm{H}^0_{\mathrm{dR}}(X) \cong \mathbb{R}$
   (b) Prove that $\mathrm{H}^1_{\mathrm{dR}}(X) \cong \mathbb{R}$ by giving a non-trivial linear map $\mathrm{Z}^1_{\mathrm{dR}}(X) \to \mathbb{R}$ by integration around a curve around the origin, and checking that the kernel is $\mathrm{B}^1_{\mathrm{dR}}(X)$ (think back to multivariate calculus).
   (c) What changes if you puncture $\mathbb{R}^2$ in k locations instead of just one?
4. The following is an exercise about the *cone* of a chain map. Let C, D be chain complexes, and let $f\colon C \to D$ be a chain map. As we discussed today, we can form a new complex from this called $\mathrm{Cone}(f)$ in the following way. Let $\mathrm{Cone}(f)_n = C_{n-1} \oplus D_n$, and define a differential by
$$d(a,b) = (-d_C(a), d_D(b) - f(a)).$$
   (a) Check that this formula really is a differential.
   (b) Prove that f is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is exact.
   (c) Prove that f is a homotopy equivalence if and only if $\mathrm{Cone}(f)$ is contractible.
5. Solve Weibel's exercise 1.5.1 and 1.5.2.
6. Let $G = C_n$ be a finite cyclic group with generator σ, and let $\mathbb{Z}G$ be the group ring of G (that is, the ring freely generated by the elements σ^k of G with multiplication given by the group operation: $n_1\sigma^{k_1} \times n_2\sigma^{k_2} = n_1 n_2 \sigma^{k_1+k_2}$. Let $\nu \in \mathbb{Z}G$ be the sum of the generators:
$$\nu = 1 + \sigma + \cdots + \sigma^{N-1}$$
   (a) Check that the differential on the chain complex
$$B_\bullet(G) = \cdots \xrightarrow{\nu} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\nu} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \longrightarrow \mathbb{Z}G$$
   squares to zero, and compute the homology of the complex.
   (b) Define
$$C^\bullet(G) = \mathrm{Hom}_{\mathbb{Z}G}(B_\bullet(G), \mathbb{Z}),$$
   meaning we take homomorphisms of $\mathbb{Z}G$-modules, and where $\mathbb{Z}$ is the module where σ acts by the identity. Compute the cohomology $\mathrm{H}^\bullet(G)$ of $C^\bullet(G)$.