Homology and Cohomology – Week 7 Exercises

Read the sections in Hatcher's *Algebraic Topology* (pages 149–153) and Bott and Tu's *Differential Forms in Algebraic Topology* (pages 19–27) on the Mayer–Vietoris sequence.

Think about these exercises, and write up solutions to three of them. If you like you can also submit exercise solutions from previous weeks.

1. Let $X = \mathbb{R}^n \setminus A$ where $A = \{(j, 0, \dots, 0) \colon j \in \mathbb{Z}_{\geq 0}\}$. Compute $\mathrm{H}_{n-1}(X)$.
2. Let S, S' be surfaces (manifolds of dimension 2). The *connected sum* of S and S', denoted $S \# S'$ is obtained by removing open neighborhoods $U \subseteq S$ and $U \subseteq S'$ homeomorphic to 2-disks and gluing along their common boundary circles:
$$S \# S' = (S \setminus U) \cup_{S^1} (S' \setminus U').$$
Compute the homology of the *oriented surface of genus* g
$$\Sigma_g = T^2 \# \cdots \# T^2$$
with $g \geq 1$ copies of T^2.
3. Compute $\mathrm{H}_\bullet(\Sigma_2, A)$ where A is a simple closed curve which:
   (a) separates Σ_2 into two genus one pieces with one boundary component each,
   (b) is a non-separating simple closed curve cutting along which gives a genus one surface with two holes, and
   (c) bounds an embedded disc.
4. Let X be a topological space, and let $n \geq 0$ be an integer. Prove that
$$\mathrm{H}_k(X \times S^n) \cong H_k(X) \oplus H_{n-k}(X)$$
for all non-negative integers k. (Hint: use induction on n and the Mayer–Vietoris sequence for a suitable cover of the n-sphere).
5. Solve Hatcher's exercise 29 on page 158.
6. Solve Hatcher's exercise 31 on page 158