Math 490 – Independent Study on Homology and Cohomology – Syllabus

Fall 2023

Instructor: Chris Elliott (pronouns: he/him)

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Office Hours: By appointment, send me an e-mail and I'll be happy to arrange a meeting outside of our normal meeting time.

What We'll Cover

Our goal will be to investigate some of the most important tools used in modern topology and geometry for the analysis and comparison of spaces – homology and cohomology groups. This falls under the general slogan of *algebraic topology*: the natural assignment of algebraic objects (like abelian groups of vector spaces) to spaces in a homotopy invariant manner, meaning that homotopy equivalent spaces should be assigned isomorphic algebraic objects. Topics we'll aim to cover will include:

- Some language of categories and functors.
- The algebra of (co)chain complexes and their (co)homology.
- Simplicial complexes and simplicial (co)homology
  - Functoriality and homotopy invariance
  - The Mayer-Vietoris sequence
- Some other recipes for computing cohomology (singular, de Rham) with statements of the equivalences between them.
- The cup product on cohomology.
- Poincaré duality.

Schedule

We will meet once a week for 90 minutes to two hours, on Fridays at 2:30pm in Chapin Hall 018. You will also meet together as a pair on another occasion during the week to discuss the readings and the exercises.

References

Each week I'll assign you some reading: chapters in a few different books. Rather than using a single textbook, you'll read a few different expositions by different authors on approximately the same material, explaining the same idea in different ways. I'll try to use references that we can find online. These books are mostly also available in the Frost Library. Some of the books we may use will include the following.

- Hatcher, *Algebraic Topology*. Available on the author's website at `https://pi.math.cornell.edu/~hatcher/AT/AT+.pdf`. This is one of the most widely used textbooks for algebraic topology. It can be a bit verbose, but the author has a nice perspective on topology, often asking you to use your geometric intuition. We might try to compare the arguments given here to more algebraic arguments in other books.
- May, *A Concise Course in Algebraic Topology*. Available on the author's website at `https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf`. May has a modern perspective that I really appreciate, but this book is very terse (the title is not kidding!). It's based on a first year graduate course at U Chicago that the author teaches that goes very fast. It can be nice to read alongside other books that go into more detail.
- Spanier, *Algebraic Topology*. Available in the Frost library. This was probably the most widely used book before Hatcher wrote his account, but these days it feels quite old-fashioned. Still, it's a classic for a reason! We'll most likely focus on chapters 4 and 5.
- Switzer, *Algebraic Topology – Homology and Homotopy*. Available in the Frost library. This is a more advanced book, and most of it goes beyond what we will cover this semester. However, we might take a look at chapter 7, which talks about an axiomatic way of characterizing homology (usually called the *Eilenberg–Steenrod theorem*).
- tom Dieck, *Algebraic Topology*. Available online at `https://www.maths.ed.ac.uk/~v1ranick/papers/diecktop.pdf`. I'm not so familiar with this book, but I've increasingly been hearing it recommended. Chapters 8–10 include a lot of the topics I want us to talk about this semester.
- Armstrong, *Basic Topology*. Available in the Frost Library. Most of this book is about point-set topology, as the name suggests, in fact we used it for Math 455 last semester. However Chapter 6 on triangulations and simplicial approximations is very clearly written and we will certainly refer to it.
- Bott and Tu, *Differential Forms in Algebraic Topology*. Available online at `https://www.math.auckland.ac.nz/~hekmati/Books/BottTu.pdf`. This book discusses a very different approach to cohomology coming from the topology of manifolds, what's called *de Rham cohomology*. This is the approach I most often use in my own research, and if you're interested I hope we'll have the chance to discuss it. This book is beautiful, and covers some deep connections between topology and real analysis.
- Riehl, *Category Theory in Context*, available on the author's website at `https://math.jhu.edu/~eriehl/context.pdf`. There wasn't a good elementary book on category theory until very recently, when two came along at once – this book and the book by Tom Leinster below. We used to have to use either Mac Lane's *Categories for the Working Mathematician* from the early 70's, which is old fashioned these days. These days I think Riehl's book is my favorite, and it is quite accessible.
- Leinster, *Basic Category Theory*, available on the arxiv at `https://arxiv.org/abs/1612.09375`. The other recent accessible category theory book. I don't think it's quite as clear as Riehl's book, but it is also very good.

Exercises and Assessment

As well as the reading, I'll give you a collection of exercises to go through each week. I'll expect you to spend time thinking about all of the problems so that we can discuss them, but you don't need to write up solutions to all of them (and it'll be fine if you get stuck on some). I will, however, expect you to type up (in LaTeX) and submit solutions to **three** problems each week, due at 6pm on Thursday by e-mail.