Math 131-H – Homework 1 – Limits

Due: in class on Tuesday September 17th.

1. Let $\log(x)$ be the natural logarithm function, with base e. In this first example, we'll use the fact that, if $x > 0$, then $\log(x) < x - 1$. We'll prove this later in the course using the derivative of $\log(x)$.

   (a) Use $\log(x) < x - 1$ to show that $1 - \frac{1}{x} < \log(x)$ if $x > 0$ (Hint: use the substitution $x = \frac{1}{u}$).

   (b) Use the squeeze theorem to find the limit

   $$\lim_{x \to 0} \frac{\log(1+x)}{x}.$$

2. In this problem we'll study a very poorly-behaved example of a function. Recall that a *rational number* is a real number that can be written as a fraction p/q, where p and q are integers. An *irrational number* is a real number which is not rational, e.g. $\sqrt{2}$. Define a function by

   $$i(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   This function is called the *indicator function of the rational numbers*, or sometimes the *Dirichlet function*. Even though $i(x)$ is defined for all real numbers, we'll show that the limit $\lim_{x \to x_0} i(x)$ does not exist for *any* value x_0.

   (a) First, show that for any real numbers $a < b$, the open interval (a, b) contains a rational number. Hint: use the fact that there exists a positive integer n such that $b - a > 1/n$, then show that there is an integer in the interval (na, nb).

   (b) If a number x is rational, then $x + \sqrt{2}$ is always irrational. Use this fact to show that for any real numbers $a < b$, the open interval (a, b) contains an *irrational* number as well.

   (c) Use the formal definition of a limit to show that, for any number x_0, the limit $\lim_{x \to x_0} i(x)$ does not exist.