Math 131-H – Homework 4 – Applications of Differentiation

Due: in class on Tuesday October 29th.

1. (a) Use implicit differentiation to compute the derivative of the function $f(x) = \operatorname{arctanh}(x)$.
   (b) Use implicit differentiation to compute the derivative of the function $g(x) = \operatorname{arccoth}(x)$.
   (c) What do you notice about your answers to parts (a) and (b)? Explain how this is possible, by discussing for which x values the functions f and g are defined.

2. In the theory of special relativity, we have the following phenomenon. Suppose an observer O at position x can see two clocks, one at rest, and one moving past them at a constant velocity v. At time t according to the stationary clock, one would expect to see the moving clock at position $x + vt$, and showing time t. Instead, the observer will see the moving clock at position

$$x_{\text{new}} = \gamma(v)(x + vt) = \frac{x + vt}{\sqrt{1 - v^2/c^2}},$$

   and showing time

$$t_{\text{new}} = \gamma(v)(t + \frac{vx}{c^2}) = \frac{t + \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}},$$

   where c is the speed of light. These formulae are called the *Lorentz transformations*, and the factor $\gamma(v)$ is called the *Lorentz factor*.

   (a) Let $\zeta = \operatorname{arctanh}(v/c)$ (sometimes called the "rapidity"). Show that the Lorentz transformations can be rewritten in the hyperbolic form

$$x_{\text{new}} = x\cosh(\zeta) + ct\sinh(\zeta)$$
$$t_{\text{new}} = t\cosh(\zeta) + \frac{x}{c}\sinh(\zeta).$$

   (b) Suppose now that the observer O starts moving too, at a constant velocity u (so, in other words, $\frac{\mathrm{d}x}{\mathrm{d}t} = u$). Use the chain rule to calculate the new apparent velocity $u_{\text{new}} = \frac{\mathrm{d}x_{\text{new}}}{\mathrm{d}t_{\text{new}}}$ in terms of u, v and the speed of light c. (Note: don't use the hyperbolic form from (a) in this part, it'll make the calculation more difficult).
   (c) Suppose $v = c$. Show that $\frac{\mathrm{d}u_{\text{new}}}{\mathrm{d}u} = 0$. In other words, if two objects are moving apart at the speed of light, it's not possible to alter their relative velocity by speeding up or slowing down.