The Toda Lattice

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In this talk I'll introduce classical integrable systems, and explain how they can arise from the data of solutions to the classical Yang-Baxter equation. I'll focus in particular on the example of the *Toda Lattice*, describing the dynamics of a cyclic molecule under a particular potential. By and large I'm following section 2.3 of [CP95], but I also referred to [STS97].

1 Completely Integrable Systems

We begin with a Hamiltonian model for classical mechanics

Definition 1.1. A *classical Hamiltonian system* is a symplectic manifold M equipped with a Hamiltonian function $H \in C^\infty(M)$.

The interpretation is that M is the *phase space* of a classical system, i.e. the space of classical states, its functions come equipped with a Poisson bracket which we can think of as *locally* giving a separation of coordinates into positions and momenta, and time evolution of functions is controlled by *Hamilton's equation*

$$\frac{df}{dt} = \{H, f\}.$$

This motivates the following definition.

Definition 1.2. A *conserved quantity* in a Hamiltonian system is a smooth function $f \in C^\infty(M)$ such that $\{f, H\} = 0$.

Remark 1.3. If you're familiar with the Lagrangian model of classical mechanics, you can derive the above setup from a Lagrangian field theory on the real line $\mathbb{R}$, i.e. from classical Lagrangian mechanics. There's a classical procedure for doing so known as the *Legendre transform*.

Definition 1.4. A classical Hamiltonian system is *completely integrable* if there exist a family of m conserved quantities $f_1, \ldots, f_m$, where $\dim(M) = 2m$, such that

1. The f_i *Poisson commute*, i.e. $\{f_i, f_j\} = 0$.
2. The f_i are *independent*, i.e. the critical points of $f = (f_i)$ as an $\mathbb{R}^m$-valued function form a measure 0 set.

One can see that m is the maximal number of commuting functions by choosing Darboux coordinates in an open patch of M.

An alternative way of looking at completely integrable systems is given by the following theorem.

Theorem 1.5 (Liouville-Arnold)**.** *The preimages of regular values of the function f in a completely integrable system are Lagrangian tori in M, and the Hamiltonian flow for H acts by constant speed rotations on these tori.*

I won't prove this, but the idea is the following. The Hamiltonian flows corresponding to the functions f_i preserve the fibres of f, and all commute, since the f_i Poisson commute. This yields an action of $\mathbb{R}^m$ on our fibres whose stabiliser one computes to be a lattice. The fact that ∇f_i span the tangent spaces and f_i Poisson commute ensures the fibres are Lagrangian. This gives an alternative definition of a completely integrable system.

Definition 1.6 (Alternative)**.** A *completely integrable system* is a symplectic manifold M with a map to $\mathbb{R}^m$ which is generically a fibration by Lagrangian tori, equipped with a function H whose Hamiltonian flow acts on the tori by constant speed rotations.

2 The Toda Lattice

Now I'll introduce my main example in the most classical way possible. I'll then rederive it using a solution to the classical Yang-Baxter equation. The example will be a Hamiltonian system designed to model a cyclic chain of particles interacting with their nearest neighbours in a particularly nice way.

Example 2.1. The *Toda lattice* is the Hamiltonian system with phase space $\mathbb{R}^{2n}$ (with its usual symplectic structure), and with Hamiltonian

$$H(p_1,\ldots,p_n;q_1,\ldots,q_n) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i-q_{i+1}} + e^{q_n-q_1}.$$

So we have a *kinetic energy* term for the particles, and a *potential* term for every neighbouring pair depending on the distance between them, growing exponentially. To show that this system is completely integrable, we'll recover it in a more abstract fashion. Specifically we'll cook up a completely integrable system from any solution to the classical Yang-Baxter equation.

2.1 Integrable Systems and the CYBE

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{R}$ equipped with a non-degenerate invariant pairing $\langle,\rangle$, and let $r \in \mathfrak{g}\otimes\mathfrak{g}$ be a solution to the CYBE. As we've already seen, r yields a Lie bialgebra structure on $\mathfrak{g}$, hence a new Lie bracket $[,]_r$ on $\mathfrak{g}$ defined by the isomorphism $\mathfrak{g}^* \to \mathfrak{g}$ given by the pairing. This Lie bracket makes $\mathfrak{g}^*$ into a *Poisson Lie group* with the bracket

$$\{f,g\}_r(\xi) = \xi\left([df_\xi, dg_\xi]_r\right)$$

where $\xi \in \mathfrak{g}^*$.

This story goes through just as well when r is instead a solution to the *modified* classical Yang-Baxter equation

$$[[r,r]] + \omega = 0$$

where $\omega = [[t,t]]$, for t the Casimir element associated to our invariant pairing.

Now, the idea is the following. We'll build an integrable system from the Poisson manifold $\mathfrak{g}^*$ with Poisson structure $\{,\}_r$. To match our definition in the previous section, we'll have to take a *symplectic leaf* of this Poisson manifold as our phase space.

Definition 2.2. A *symplectic leaf* in a Poisson manifold is a maximal submanifold in which the Poisson structure restricts to a symplectic structure. Equivalently, a symplectic leaf is an equivalence class under the equivalence relation $x \sim y$ if there is a piecewise smooth path from x to y whose smooth segments are trajectories of Hamiltonian vector fields.

Example 2.3. In $\mathfrak{g}^*$ with either its standard Poisson structure or the Poisson structure given $\{,\}_r$, the symplectic leaves are exactly the coadjoint orbits (where the coadjoint action is defined using the appropriate Lie bracket for the Poisson structure).

Now, here's the important result.

Proposition 2.4. Functions $f_1, \dots, f_k$ on $\mathfrak{g}^*$ satisfy and $\{f_i, f_j\} = 0$ and $\{f_i, f_j\}_r = 0$ whenever they are coadjoint invariant (with respect to the ordinary coadjoint action.

In particular, if we set $f_1 = H$, then restricting to a symplectic leaf for $\{,\}_r$ we end up with k Poisson commuting conserved quantities, and if we can find enough independent coadjoint invariant functions then we'll have a completely integrable system.

Proof. It's just a short calculation. I'll just prove the latter. Firstly, we can write the Lie bracket $[,]_r$ as

$$[X, Y]_r = [\rho(X), Y] + [X, \rho(Y)]$$

where ρ is the map $\mathfrak{g} \to \mathfrak{g}$ given by precomposing the map $r\colon \mathfrak{g}^* \to \mathfrak{g}$ by the map given by the pairing. Let f and g be coadjoint invariant functions. Then

$$\begin{aligned} \{f, g\}_r(\xi) &= \xi([df_\xi, dg_\xi]_r) \\ &= \xi([\rho(df_\xi), dg_\xi]) + \xi([df_\xi, \rho(dg_\xi)]) \\ &= (ad^*_{\rho(df_\xi)}(\xi))(dg_\xi) - (ad^*_{\rho(dg_\xi)}(\xi))(df_\xi) \end{aligned}$$

where the last line indicates the coadjoint action of an element of $\mathfrak{g}$ on $\mathfrak{g}^*$. Infinitesimal coadjoint invariance of f and g implies that

$$(ad^*_X(\xi))(df_\xi) = 0$$

for all $X \in \mathfrak{g},\ \xi \in \mathfrak{g}^*$. □

Now, we might ask why it was meaningful that we had a solution to the (modified) CYBE. The abstract reason is that such solutions produce *Lax pairs* for a Hamiltonian system. These give a very nice form to Hamilton's equations which allow one to easily generate conserved quantities. I'll give the definition, followed a generalisation of the above proposition.

Definition 2.5. A *Lax pair* for a Hamiltonian system is a pair of smooth $\mathfrak{g}$-valued functions L and P on $M \times \mathbb{R}$ satisfying the equation

$$\frac{dL}{dt} = [L, P].$$

The existence of a Lax pair allows one to generate conserved quantities from any adjoint-invariant polynomial function $f \in \mathcal{O}(\mathfrak{g})$; the function $f \circ L$ is then conserved.

Proposition 2.6. A Hamiltonian system on a symplectic leaf in $(\mathfrak{g}^*, \{,\}_r)$ where the Hamiltonian is coadjoint invariant admits a Lax pair whenever r is a solution to the modified CYBE. Moreover

$$\{L, L\}_r = [r, L \otimes 1, 1 \otimes L].$$

The operator L here is just the map $\mathfrak{g}^* \to \mathfrak{g}$ given by the pairing, and the operator P is given by

$$P(\xi) = \rho(dH(\xi)).$$

2.2 Return to the Toda Lattice

Example 2.7. Okay, now let's describe an actual integrable system. Set $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{R})$. We'll use the solution

$$r' = \left(\sum_{1 \le i < j \le n+1} E_{i,j} \otimes E_{j,i} - E_{j,i} \otimes E_{i,j} \right) + t$$

to the CYBE, where $E_{i,j}$ is the matrix with a 1 in the i,j position, and 0 elsewhere, and where t is the Casimir (this is the same as the solution Kevin discussed last time). This yields a solution

$$r = \sum_{1 \le i < j \le n+1} E_{i,j} \otimes E_{j,i} - E_{j,i} \otimes E_{i,j}$$

to the *modified* CYBE.

The symplectic leaf we'll use will be denoted $\mathfrak{g}_T^* \subseteq \mathfrak{g}^*$, and is the image under the isomorphism given by the pairing of the linear span of elements

$$\{H_i = E_{i,i} - E_{i+1,i+1},\ X_i = E_{i,i+1} + E_{i+1,i} :\ i = 1, \dots n\}$$

in $\mathfrak{g}$. Write h_i, x_j for the basis of $\mathfrak{g}_T^*$ dual to the above elements. Note that

$$\rho(H_i) = 0, \quad \rho(E_{i,j}) = \pm\frac{1}{2} E_{j,i} \quad i \neq j$$

where the sign depends on whether i or j is greater. This allows one to check that $\mathfrak{g}_T^*$ is a coadjoint orbit, and therefore a symplectic leaf.

We can also see that the system, with any coadjoint invariant function as Hamiltonian, is completely integrable using our proposition. This follows by counting the rank of the algebra of coadjoint invariant functions on $\mathfrak{g}^*$. This rank is independent of r, so is computed (by considering the dense subalgebra of polynomial functions on $\mathfrak{g}^*$) as the rank of

$$\begin{aligned} \mathcal{O}(\mathfrak{g}^*)^G &\cong \mathcal{O}(\mathfrak{g})^G \\ &\cong \mathcal{O}(\mathfrak{h})^W \\ &\cong \mathbb{R}[x_1, \dots, x_n]^{S_n} \\ &\cong \mathbb{R}[y_1, \dots, y_n] \end{aligned}$$

thus there are n independent coadjoint invariant functions, and choosing any generating set makes our Hamiltonian system completely integrable.

A natural choice of coadjoint invariant function to play the role of Hamiltonian is the function

$$H(\xi) = \frac{1}{2}\langle \xi, \xi \rangle.$$

Claim. This Hamiltonian system recovers the Toda lattice described above. Therefore the Toda lattice is a completely integrable system.

Proof. We choose a new set of coordinates for the open subset of $\mathfrak{g}_T^*$ where all x coordinates are positive (matrices in this subset are called *Jacobi matrices*), setting $p_i = h_i$, and

$$x_i = e^{q_i - q_{i+1}},$$

then observing that we recover the Hamiltonian

$$H(p_1, \dots, p_n; q_1, \dots, q_n) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + e^{q_n - q_1}$$

described above. The original basis $\{h_i, x_j\}$ is sometimes called a set of *Flashke variables* for the system. ☐

Remark 2.8. 1. We can explicitly describe some coadjoint invariant, hence conserved quantities. A good family of examples is given by

$$f_k(\xi) = \mathrm{Tr}(\kappa(\xi)^k)$$

where $\kappa\colon \mathfrak{g}^* \to \mathfrak{g}$ is the isomorphism given by the pairing. These will be independent for $k = 2, \ldots n+1$.

2. This discussion generalises, allowing us to define Toda lattice like integrable systems associated to any real split semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. We'd define it as follows. Choose a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$, and filter $\mathfrak{b}$ by setting $\mathfrak{b}^{(i)} = [\mathfrak{b}^{(i-1)}, \mathfrak{b}^{(i-1)}]$. Define a Lie subalgebra $\mathfrak{s} = \mathfrak{b}/\mathfrak{b}^{(2)}$. This exponentiates to a Lie group S. Define $\mathfrak{g}_T^*$ to be an open orbit for the coadjoint action of S on $\mathfrak{s}^* \subseteq \mathfrak{g}^*$. It is spanned by the Cartan, and by positive multiples of $e_\alpha + e_{-\alpha}$ for positive roots α, where e_α lies in the root space.

References

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