Vacua and Singular Supports

Chris Elliott

May 15th 2017

Abstract

The notion of singular support for coherent sheaves was introduced by Arinkin and Gaitsgory in order to carefully state the geometric Langlands conjecture. This is a conjectural equivalence of categories of sheaves on certain moduli spaces: in order to make the conjecture reasonable one needs to restrict to sheaves which satisfy a certain ”singular support condition”. In this talk I'll explain how to think about this singular support condition from the point of view of boundary conditions in twisted $N = 4$ gauge theory. Specifically, Arinkin and Gaitsgory's singular support condition arises by considering only those boundary conditions which are compatible with a natural choice of vacuum state. By allowing this vacuum state to move away from this natural choice we see aspects of a rich additional structure for the geometric Langlands correspondence. This work is joint with Philsang Yoo.

1 Introduction

Today I'll explain some joint work with Philsang Yoo which attempts to explain the modern statement of the geometric Langlands conjecture in terms of supersymmetric field theory. In the pioneering work of Kapustin and Witten [KW06] the geometric Langlands correspondence occurs as an equivalence between categories of boundary conditions along a Riemann surface Σ in S-dual $N = 4$ supersymmetric field theories, and in dual topological twists. The equivalence they explain is closely related to the “best hope” geometric Langlands conjecture as formulated by Beilinson and Drinfeld, which says the following.

1.1 Geometric Langlands

Let G be a reductive algebraic group over $\mathbb{C}$, and let Σ be a smooth compact complex curve. Let $\mathrm{Bun}_G(\Sigma)$ be the moduli stack of algebraic G-bundles on Σ, and let $\mathrm{Flat}_G(\Sigma)$ be the moduli stack of G-bundles with flat connection on Σ.

Conjecture 1.1 (“Best Hope” version of Geometric Langlands)**.** There is an equivalence of categories

$$\text{D-mod}(\mathrm{Bun}_G(\Sigma)) \cong \mathrm{QC}(\mathrm{Flat}_{G^\vee}(\Sigma))$$

where $G^\vee$ is the Langlands dual group of G.

Remark 1.2. There's a lot to say about this conjecture: one doesn't just expect an equivalence, but an equivalence compatible with all sorts of extra structure. One way to think about it is as an extremely fancy version of *Fourier duality* (non-abelian, categorified). You can think of the space $\mathrm{Flat}_{G^\vee}(\Sigma)$ as parameterizing a nice family of D-modules on $\mathrm{Bun}_G(\Sigma)$: those that are eigenobjects for a family of natural symmetries. Points in $\mathrm{Flat}_{G^\vee}(\Sigma)$ parameterize these eigenobjects in the sense that the above equivalence sends an eigenobject to a skyscraper sheaf at a point in $\mathrm{Flat}_{G^\vee}(\Sigma)$ (its “eigenvalue”). When G is abelian the equivalence is *literally* given by a Fourier transform (more precisely a twisted Fourier-Mukai transform). This is a theorem of Laumon [Lau96] and Rothstein [Rot96].

As stated above this conjecture is only true when G is abelian. There's a basic reason: one of the structures that the equivalence should respect is the *geometric Eisenstein functors* on the left- and right-hand sides. When you have a parabolic subgroup $P \subseteq G$ with Levi L there is a pull-push functor from D-mod($\mathrm{Bun}_L(\Sigma)$ to D-mod($\mathrm{Bun}_G(\Sigma)$) (and similarly on the right-hand side). While there is a good 6-functor formalism for D-modules on stacks it turns out that for quasi-coherent sheaves on derived stacks the pullback is badly-behaved – it doesn't preserve compact objects. Because these geometric Eisenstein functors are well-behaved on one side but not the other, there's no way the equivalence as stated above can hold.

More concretely, V. Lafforgue [Laf09] proved the geometric Langlands conjecture for the curve $\Sigma = \mathbb{CP}^1$. He showed that D-mod($\mathrm{Bun}_G(\mathbb{CP}^1)$) was equivalent to a category of sheaves strictly larger than QC($\mathrm{Flat}_{G^\vee}(\mathbb{CP}^1)$). Arinkin and Gaitsgory [AG12] demonstrated how to extend Lafforgue's answer to general curves Σ, fixing the technical issue with pull-back functors, and therefore give a plausible statement of the geometric Langlands correspondence. They enlarged QC($\mathrm{Flat}_{G^\vee}(\Sigma)$) to a larger category where the pullback functors associated to inclusions $L \hookrightarrow P$ preserved compact objects in the smallest possible way, which recovers Lafforgue's answer for $\Sigma = \mathbb{CP}^1$.

Conjecture 1.3 (Arinkin and Gaitsgory's version of Geometric Langlands)**.** There is an equivalence of categories

$$\text{D-mod}(\mathrm{Bun}_G(\Sigma)) \cong \mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_{G^\vee}(\Sigma))$$

where $\mathrm{IndCoh}_{\mathcal{N}_G^\vee}(\mathrm{Flat}_{G^\vee}(\Sigma))$ is the category of those ind-coherent sheaves on $\mathrm{Flat}_{G^\vee}(\Sigma)$ (that is, objects in the ind-completion of the category of coherent sheaves) which satisfy a condition called "nilpotent singular support".

Before I tell you what nilpotent singular support means, let me quickly detour to talk about the connection to gauge theory, and the work of Kapustin and Witten.

1.2 Kapustin-Witten Theories

Let's recall Kapustin and Witten's story. Start out with $N = 4$ super Yang-Mills theory in dimension 4. Kapustin and Witten constructed a $\mathbb{CP}^1$-family of topological twists of this theory, and demonstrated that if you dimensionally reduce these theories at the points $(0 : 1)$ and $(1 : 0)$ respectively you obtain the B-model with target $\mathrm{Loc}_G(\Sigma)$ and the A-model with target $\mathrm{Bun}_G(\Sigma)$. Here $\mathrm{Loc}_G(\Sigma)$ is the stack of representations of the fundamental group of Σ into G; this is analytically equivalent to $\mathrm{Flat}_G(\Sigma)$ but crucially *not* algebraically equivalent, which is why I give it a different name [1].

Kapustin and Witten argue that the geometric Langlands conjecture is a consequence of S-duality for $N = 4$ supersymmetric field theories. This follows from two observations.

1. Firstly, S-duality exchanges the gauge group G and its Langlands dual $G^\vee$, and it acts antipodally on the $\mathbb{CP}^1$ of twisted theories. In particular the topologicall twisted theories at the points $(0 : 1)$ and $(1 : 0)$ are S-dual.
2. The category of boundary conditions along Σ in the B-twisted theory for gauge group $G^\vee$ is equivalent to QC($\mathrm{Flat}_{G^\vee}(\Sigma)$). Dually, Kapustin and Witten argue that the category of boundary conditions in the A-twisted theory for gauge group G is equivalent to the category D-mod($\mathrm{Bun}_G(\Sigma)$) of D-modules. S-duality implies that these two categories are equivalent.

There are two issues with this argument if one wishes to truly draw a bridge with the geometric Langlands conjecture. Firstly, the issue of algebraic structure. Geometric Langlands in its usual form involves sheaves on the *algebraic* stack $\mathrm{Flat}_{G^\vee}(\Sigma)$, but this algebraic structure isn't visible in Kapustin and Witten's argument (which is purely topological, so for instance only depends on Σ as a smooth manifold, not an algebraic curve. In joint work with Philsang Yoo [EY15] we explain how one can reintroduce these algebraic structures on moduli spaces into the physical story.

[1] However, take care, in the geometric representation theory what I call $\mathrm{Flat}_G(\Sigma)$ is frequently called $\mathrm{Loc}_G(\Sigma)$ or $\mathrm{LocSys}_G(\Sigma)$.

Secondly, and the topic for today's talk, Kapustin and Witten only discuss the best hope form of the geometric Langlands conjecture, which as we've seen is generally false. Today, I'll explain how to think about the Arinkin-Gaitsgory correction to the geometric Langlands conjecture from the physical point of view, in terms of fixing a choice of vacuum for the twisted theory.

2 Singular Support Conditions

I'll now explain what it means to restrict the "singular support" of a coherent sheaf, then in the next section I'll explain the connection to a general story in TQFT. The definition of singular support that I'll give is due to Arinkin and Gaitsgory [AG12], based on a general story about support conditions in a dg-category developed by Benson, Iyengar and Krause [BIK08].

I'll start with a very general and abstract definition that makes sense for any dg-category $\mathcal{B}$ (all categories from now on will be dg-categories). We say a monoidal category $\mathcal{L}$ *acts* on a category $\mathcal{B}$ if there's a monoidal functor from $\mathcal{L}$ to $\mathrm{End}(\mathcal{B})$.

Definition 2.1. The *Hochschild cochains* of a dg-category $\mathcal{B}$ are the algebra $\mathrm{HC}^\bullet(\mathcal{B})$ of natural transformations from $\mathrm{id}_\mathcal{B}$ to itself. There's a canonical action of the monoidal category $\mathrm{HC}^\bullet(\mathcal{B})$-mod. In fact this action is universal: any action of a monoidal category $\mathcal{L}$ on $\mathcal{B}$ factors through a monoidal functor to $\mathrm{HC}^\bullet(\mathcal{B})$-mod.

Example 2.2. The physical origin of this sort of structure will be the action of the local observables in a quantum field theory on the category of boundary conditions. I'll restrict attention to topological field theories. In a topological field theory the local observables form an $\mathbb{E}_n$-algebra and there's a category of boundary conditions along any manifold M of codimension 2 admitting an action of this $\mathbb{E}_n$-algebra. We think about this action as follows. Let $\mathcal{B}$ be the category of boundary conditions along M in a topological field theory, and let $\mathcal{F}$ be an object in $\mathcal{B}$. The algebra $\mathrm{End}_\mathcal{B}(\mathcal{F})$ describes the space of local observables in the *bulk-boundary theory* associated to think boundary condition, and there's a tautological map from bulk observables into bulk-boundary observables for any choice of boundary condition.

Let's restrict attention to the example where $\mathcal{B}$ is the category $\mathrm{IndCoh}(X)$ of ind-coherent sheaves on something. Eventually X will be a derived stack, but for now let's just take it to be an affine derived scheme (that is, Spec of a commutative dga). The *singular support* of a sheaf $\mathcal{F}$ in $\mathrm{IndCoh}(X)$ is – roughly speaking – its support in the even Hochschild cohomology $\mathrm{HH}^{2\bullet}(X)$ with respect to the action we described above (the reason we only take even cohomology is to make sure we get a commutative rather than supercommutative ring). More precisely Arinkin and Gaitsgory define singular support with respect to the action of an algebra called $\mathcal{O}(\mathrm{Sing}(X))$ that maps into $\mathrm{HH}^{2\bullet}(X)$. We define this algebra as follows.

Definition 2.3. We say X is *quasi-smooth* if its tangent complex is concentrated in degrees ≤ 1 (recall that being smooth is equivalent to have tangent complex in degrees ≤ 0). If X is quasi-smooth then its *scheme of singularities* is the classical part of the -1-shifted cotangent space

$$\mathrm{Sing}(X) = (T^*[-1]X)^{\mathrm{cl}}.$$

Concretely, for an affine derived scheme $\mathrm{Sing}(X)$ is the ordinary affine scheme whose functions are generated by $H^1(T_X) = H^0(T_X[1])$ as an $\mathcal{O}(X)$-module.

Now, this is closely related to the Hochschild cohomology of X by the following fact.

Theorem 2.4 (Hochschild-Kostant-Rosenberg)**.** Let X be an (eventually coconnective) affine derived scheme. There is a canonical filtration on $\mathrm{HC}^\bullet(X)$ whose associated graded is $\mathcal{O}(T^*[1]X)$.

I don't want to spell out all the details, but one can use this fact to define a map from $\mathcal{O}(\mathrm{Sing}(X))$ to $\mathrm{HC}^\bullet(X)$. In particular the category $\mathrm{QC}(\mathrm{Sing}(X))$ of $\mathcal{O}(\mathrm{Sing}(X))$-modules acts on the category $\mathrm{IndCoh}(X)$. The map involves a shift by two because $\mathcal{O}(\mathrm{Sing}(X))$ was built from the -1-shifted cotangent space whereas $\mathrm{HC}^\bullet(X)$ is related to the $+1$-shifted cotangent space. We'll mention this shift again a bit later.

We can now define the category with a singular support condition. Let Y be a closed subset of $\mathrm{Sing}(X)$.

Definition 2.5. The category of ind-coherent sheaves on X with *singular support* in Y is defined to be the localization

$$\mathrm{IndCoh}_Y(X) = \mathrm{IndCoh}(X) \otimes_{\mathrm{QC}(\mathrm{Sing}(X))} \mathrm{QC}(\mathrm{Sing}(X))_Y$$

where $\mathrm{QC}(\mathrm{Sing}(X))_Y$ is the category of sheaves with set-theoretic support contained in Y.

Remark 2.6. If X is not just an affine derived scheme but a more general derived stack then you can still define the category of shcaves with singular support in a closed subset Y of $\mathrm{Sing}(X)$ as a limit over all smooth maps $Z \to X$ from affine derived schemes into X.

Example 2.7. From the point of view of quantum field theory, the category $\mathrm{IndCoh}(X)$ describes a completed version of the category of boundary conditions in the 2d B-model with target X. Singular support conditions will admit a nice physical description from this point of view which we'll explain shortly – one should interpret the algebra $\mathcal{O}(\mathrm{Sing}(X))$ as the algebra of local operators in the 2d B-model (up to one of these ubiquitous degree shifts by two).

So I've nearly explained to you what the category $\mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma))$ is. All I have to do is tell you what the subset $\mathcal{N}_{G^\vee} \subseteq \mathrm{Sing}(\mathrm{Flat}_{G^\vee}(\Sigma))$ is. First I'll describe the space of singularities of $\mathrm{Flat}_{G^\vee}(\Sigma)$. This is a straightforward computation using the fact that the tangent complex to $\mathrm{Flat}_{G^\vee}(\Sigma)$ is the de Rham complex of Σ with coefficients in $(\mathfrak{g}^\vee)^*$, shifted down in cohomological degree by one.

Proposition 2.8. The space $\mathrm{Sing}(\mathrm{Flat}_{G^\vee}(\Sigma))$ is the (classical) moduli stack whose closed points are triples (P, ∇, ϕ) where (P, ∇) is a classical point of $\mathrm{Flat}_{G^\vee}(\Sigma)$ (i.e. a $G^\vee$-bundle with flat connection) modulo gauge transformations, and ϕ is a flat section

$$\phi \in H^0_\nabla(\Sigma; (\mathfrak{g}^\vee)^*_P)$$

of the coadjoint bundle of P. We call this space $\mathrm{Arth}_{G^\vee}(\Sigma)$ – the stack of $G^\vee$-*Arthur parameters* of Σ.

Definition 2.9. The *global nilpotent cone* $\mathcal{N}_{G^\vee} \subseteq \mathrm{Arth}_{G^\vee}(\Sigma)$ is the substack consisting of Arthur parameters (P, ∇, ϕ) where the value ϕ_x of ϕ at a point $x \in \Sigma$ is *nilpotent* as an element of the dual Lie algebra $(\mathfrak{g}^\vee)^*$. This condition doesn't depend on the choice of point x because the section ϕ was a flat section.

So now we know what the Arinkin-Gaitsgory category $\mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma))$ means. It's the category of ind-coherent sheaves whose singular support lies in the global nilpotent cone. My goal for the remainder of this talk will be to explain how this condition arises from quantum field theory.

3 Localization for TQFTs

3.1 The General Story

So what's the connection between singular support conditions and quantum field theory? The idea we'll take advantage of is that every quantum field theory has the structure of a module over its algebra of local observables. We'll restrict to the example of topological field theories, so suppose $\mathcal{B}$ is the category of boundary conditions in an n-dimensional TQFT along a compact $(n-2)$-manifold M. As we mentioned above, this category is a module for the algebra A of local quantum observables in our field theory.

Definition 3.1. *States* in a quantum field theory on $\mathbb{R}^n$ with algebra $\mathrm{Obs}^{\mathrm{q}}(B^n)$ of local observables are functionals $\phi \colon \mathrm{Obs}^{\mathrm{q}}(B^n) \to \mathbb{R}$. A state ϕ is a *vacuum state* if it translation invariant and satisfies the *cluster decomposition property*, which says $\mathcal{O}_1$ on $B_{r_1}(0)$ and $\mathcal{O}_2$ on $B_{r_2}(0)$, we have

$$(\mathcal{O}_1 * \tau_x(\mathcal{O}_2))(\phi) - \mathcal{O}_1(\phi)\mathcal{O}_2(\phi) \to 0 \text{ as } x \to \infty$$

where τ_x denotes the translation of an observable by $x \in \mathbb{R}^n$. In a *topological* field theory this just says that ϕ is a *ring homomorphism*, so vacuum states are nothing but points in the spectrum $\mathrm{Spec}(\mathrm{Obs}^{\mathrm{q}}(B^n))$.

So, let $v \in \operatorname{Spec} A$ be a vacuum state. An object $\mathcal{F} \in \mathcal{B}$ is *supported* at v if when we *localize* the algebra $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ at the point v the result is non-zero. If this localization doesn't vanish, that means precisely that there are some local bulk-boundary observables that can be defined on a sufficiently small neighbourhood of v. We say that the localized observables are *compatible* with the vacuum v, and if the localization doesn't vanish we say the boundary condition $\mathcal{F}$ is compatible with v.

3.2 Localization in Kapustin-Witten Theories

Let's apply this to a specific example coming from Kapustin and Witten's family of topological twists. We'll focus on the B-twist and explain how we see Arinkin-Gaitsgory's nilpotent singular support condition by looking at the category of boundary conditions compatible with a specific choice of vacuum.

Remark 3.2. Because I'll be working with only one side of the Langlands correspondence at a time I'll describe the B-twist with gauge group G, not $G^\vee$. This keeps the notation a bit less cluttered.

First we'll compute the moduli space of vacua in the B-twisted $N = 4$ theory. This is a straightforward calculation: the classical algebra of observables is just the algebra of functions on the space of solutions to the equations of motion on a ball. In the B-twisted theory this space is $T^*[3]BG \cong \mathfrak{g}^*[2]/W$. The algebra $\mathcal{O}(\mathfrak{g}^*[2]/G)$ is equivalent to $\mathcal{O}(\mathfrak{h}^*[2]/W)$ where $\mathfrak{h}$ is a Cartan subalgebra and W is the Weyl group (that is, $\mathfrak{h}^*[2]/W$ is the affinization of $\mathfrak{g}^*[2]/G$ – concretely the map $\mathfrak{g}^*[2]/G \to \mathfrak{h}^*[2]/W$ is the map sending a conjugacy class of matrix to its eigenvalues).

It turns out that this algebra $\mathcal{O}(\mathfrak{h}^*[2]/W)$ is exact at the quantum level. To check this one needs to check that it doesn't admit any $\mathbb{P}_4$ deformations, then use Kontsevich's formality result. So our algebra A of local quantum observables is just $\mathcal{O}(\mathfrak{h}^*[2]/W)$ as a graded commutative algebra.

Remark 3.3. There's a trick that allows us to work with the unshifted algebra $\mathcal{O}(\mathfrak{h}^*/W)$ instead. Roughly speaking, when we define the topological twist we construct a family of theories over a fermionic degree 1 parameter t. If we keep track of this parameter we actually obtain an action of a algebra of twisted local observables $\mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)$ including this parameter. If we restrict to H^0 of this algebra we get $\mathcal{O}(\mathfrak{h}^*/W)$ with no shift.

Naïvely, we describe the category of boundary conditions in the B-twisted theory by $\mathrm{Coh}(\mathrm{Flat}_G(\Sigma))$, or rather by its completion $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$, as in the usual description of the B-model. The action of local observables on this category has a nice description in terms of geometric representation theory. If we choose a point $x \in \Sigma$ the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ becomes a module for the category of *line operators*, which is given by

$$\mathcal{L} = \mathrm{IndCoh}(\mathrm{Flat}_G(\mathbb{B}))$$

where $\mathbb{B} = \mathbb{D} \cup_{\mathbb{D}^\times} \mathbb{D}$ is the "formal bubble" obtained by gluing two formal disks together along a formal punctured disk. This monoidal category acts by convolution – double a formal neighbourhood of the point $x \in \Sigma$ and pull-tensor-push along the diagram

$$\begin{tikzcd}
& \mathrm{Flat}_G(\mathbb{B}) & \\
& \mathrm{Flat}_G(\Sigma \amalg_{\mathbb{D}^\times} \mathbb{D}_x) \arrow[u, "q_x"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Flat}_G(\Sigma) & & \mathrm{Flat}_G(\Sigma).
\end{tikzcd}$$

In geometric representation theory the category $\mathcal{L}$ is called the "spectral Hecke category". The monoidal unit of $\mathcal{L}$ is given by the skyscraper sheaf at the trivial bundle. If one computes its endomorphism algebra in $\mathcal{L}$ one sees our algebra A of local operators:

$$\mathrm{End}_{\mathcal{L}}(\delta_1) \cong \mathcal{O}(\mathfrak{h}^*[2]/W) = A.$$

This isn't so surprising – morphisms between two line operators should be given by states on a strip compatible with these line operators on two sides, and if the line operator on both sides this just gives all states, or all local operators under the state-operator correspondence.

The upshot to all this is that we obtain our action of the algebra A of local operators as follows.

$$\begin{aligned} &\text{The action defines a functor } \mathcal{L} \to \mathrm{End}(\mathcal{B}) \\ &\quad \text{which induces a map } A = \mathrm{End}_{\mathcal{L}}(\delta_1) \to \mathrm{End}_{\mathrm{End}_{\mathcal{B}}}(\mathrm{id}_{\mathcal{B}}) = \mathrm{HC}^\bullet(\mathcal{B}) \\ &\quad \text{and therefore a map } A \to \mathrm{End}_{\mathcal{B}}(\mathcal{F}) \end{aligned}$$

for each object $\mathcal{F}$ by the universal property of Hochschild cochains. The second line came from the first line by applying the functor to the algebra of endomorphisms of the unit on each side.

We can now state the main result.

Theorem 3.4 (E-Yoo)**.** The full category of boundary conditions compatible with the vacuum $0 \in \mathfrak{h}^*[2]/W$ is equivalent to Arinkin and Gaitsgory's nilpotent singular support category $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$.

There's a simple reason that leads us to expect such a result. There's a natural map – evaluation at a point $x \in \Sigma$ from $\mathrm{Arth}_G(\Sigma)$ to $\mathfrak{g}^*/G$. Post-composing with the eigenvalue map defines a map $\mathrm{Arth}_G(\Sigma) \to \mathfrak{h}^*/W$. The action of $\mathcal{O}(\mathfrak{h}^*[2]/W)$ on the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ factors through the natural action of $\mathcal{O}(\mathrm{Arth}_G(\Sigma))$ by which we define singular support, by pullback along the eigenvalue map (note that this is only a map of *ungraded* commutative rings, one needs to be more careful to keep track of all the shifts by two). What's more, the global nilpotent cone can by thought of as coming from the following pullback:

$$\begin{tikzcd} \mathcal{N}_G \arrow[r] \arrow[d] & \mathrm{Arth}_G(\Sigma) \arrow[d, "\mathrm{ev}_x"] \\ \{0\} \arrow[r] & \mathfrak{h}^*/W, \end{tikzcd}$$

which means being supported at 0 in $\mathfrak{h}^*/W$ is equivalent to being supported on $\mathcal{N}_G$ in $\mathrm{Arth}_G(\Sigma)$.

There's actually a bit more we can say. By replacing $\mathcal{O}(\mathfrak{h}^*[2]/W)$ by its shifted version $\mathcal{O}(\mathfrak{h}^*/W)$ it makes sense to ask for the category of boundary conditions compatible with *any* vacuum $v \in \mathfrak{h}^*/W$. We can compute this category by a similar method to the one I just described, and we conjecture that the results fit together in a nice way: that the categories one obtains are equivalent to Arinkin-Gaitsgory categories with the symmetry group broken to a subgroup compatibly with the vacuum. We conjecture the following (and we have some evidence supporting the conjecture).

Conjecture 3.5 (Gauge symmetry breaking)**.** The full subcategory of objects in $\mathrm{IndCoh}(\mathrm{Loc}_G(\Sigma))$ compatible with the vacuum $v \in \mathfrak{h}^*/W$ is equivalent to $\mathrm{IndCoh}_{\mathcal{N}_L}(\mathrm{Loc}_L(\Sigma))$, where $L \subseteq G$ is the stabilizer of v in $\mathfrak{g}^*$.

References

[AG12] Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves, and the geometric Langlands conjecture. *arXiv preprint arXiv:1201.6343*, 2012.

[BIK08] Dave Benson, Srikanth Iyengar, and Henning Krause. Local cohomology and support for triangulated categories. In *Annales scientifiques de lÉNS*, volume 41, pages 575–621, 2008.

[EY15] Chris Elliott and Philsang Yoo. Geometric Langlands twists of $N = 4$ gauge theory from derived algebraic geometry. *arXiv preprint arXiv:1507.03048*, 2015.

[KW06] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *arXiv preprint hep-th/0604151*, 2006.

[Laf09] Vincent Lafforgue. Quelques calculs reliés à la correspondance de Langlands géométrique pour $\mathbb{P}^1$ (version provisoire). Available at author's website: `http://www.math.jussieu.fr/~vlafforg`, 2009.

[Lau96] Gerard Laumon. Transformation de Fourier généralisée. *arXiv preprint alg-geom/9603004*, 1996.

[Rot96] Mitchell Rothstein. Sheaves with connection on abelian varieties. *Duke Mathematical Journal*, 84(3):565–598, 1996.

Institut des Hautes Études Scientifiques
35 Route de Chartres, Bures-sur-Yvette, 91440, France
celliott@ihes.fr