INSTANTON MODULI AND COMPACTIFICATION

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(1) Instanton (definition)
(2) ADHM construction
(3) Compactification

1. Instantons

1.1. **Notation.** Throughout this talk, we will use the following notation:

G: a (semi-simple) Lie group, typically $SU(n)$
$\mathfrak{g}$: its Lie algebra
X: a (simply connected) 4-fold (typically, S^4)
P: a principal G-bundle $\pi : P \to X$
A: a connection on P
F_A: its curvature
$\Omega^p_X(\mathfrak{g})$: differential p-forms on X with values in $\mathfrak{g}$, i.e., $\Omega^1_X(\mathfrak{g}) := \Gamma(T^*X \otimes \mathfrak{g})$.

1.2. **Primer on Connections.** Recall from Peng's talk that a **connection** on $P \to X$ can be thought of in three ways:

(1) As a field of horizontal subspaces: $T_pP = H_pP \oplus V_pP$, where $VP = \ker(\pi_*)$. (Note that $V_p \cong \mathfrak{g}$, and $H_p \cong T_{\pi(p)}X$).
(2) As a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_X(\mathfrak{g})$ which is invariant w.r.t. the induced G-action on $\Omega^1_X(\mathfrak{g})$.
(3) Given a representation of G on W, as a vector bundle connection on the associated bundle $E := P \times_G W \to X$

$$\nabla_A : \Omega^0_X(E) \to \Omega^1_X(E),$$

where ∇_A is a linear map satisfying the Leibniz rule: $\nabla(f \cdot s) = f \cdot \nabla s + df \cdot s$ for $f \in C^\infty(X)$, $s \in \Gamma(X, E)$.

The first two of these definitions are seen to be equivalent by setting $H_pP = \ker A_p$ for (2)→(1), or $A_p = T_pP \to V_pP$ (projection) for (1)→(2). For (3), we think of P as being the frame bundle for E, and then describe a horizontal frame of sections.

For concreteness, we will mostly use (3) in this talk, i.e., fix a vector bundle $E \to X$, and then think of P as the frame bundle of E. However, everything can still be done for principal bundles, too.

From ∇_A, we can build a new operator

$$d_A : \Omega^k_X(E) \to \Omega^{k+1}_X(E)$$

by requiring that $d_A = \nabla_A$ for sections of E and $d_A(\omega \wedge \theta) = (d_A\omega) \wedge \theta + (-1)^{|\omega|}\omega \wedge d_A\theta$. In general, $d_A^2 \neq 0$, and we give this a special name: the **curvature**

$$F_A := d_A^2 : E \to \Omega^2_X(E)$$

Locally,

$$d_A = d + A\wedge,$$

where $A \in \Omega^1_X(E)$ is an E-valued 1-form, and

$$F_A = dA + A \wedge A \in \Omega^2_X(E)$$

The covariant derivative along $v \in TX$ of a section s is given by $\iota_v d_A s$.

If X has the additional structure of a Riemannian metric, the formal adjoint d_A^* to d_A can be defined:

$$\int_X \langle d_A\phi, \psi\rangle\, d\mu = \int_X \langle \phi, d_A^*\psi\rangle\, d\mu.$$

If in addition $\dim X = 4$, then from Hodge theory, 2-forms on X decompose into self-dual and anti-self-dual parts. This extends to E-valued forms

$$\Omega^2_X(E) = \Omega^+_X(E) \oplus \Omega^-_X(E),$$

so

$$F_A = F_A^+ + F_A^-.$$

A connection A is called **anti-self-dual** (ASD) if $F_A^+ = 0$ (i.e., $\star F_A = -F_A$).

1.3. **Yang-Mills Theory.** Yang-Mills theory is a field theory defined for principal G bundles $\pi : P \to X$. The fields of the theory are connections, and the action is (up to some constants)

$$S(A) := \int_X |F_A|^2\, d\mu. \tag{1.1}$$

S is conformally invariant in dimension 4: if $g \mapsto cg$ is a conformal transformation, then $d\mu \mapsto c^d d\mu$ and $F_A \mapsto c^{-2}F_A$, so for $\dim X = d = 4$,

$$\int_X c^{4-4}|F_A|^2\, d\mu = \int_X |F_A|^2\, d\mu.$$

For a G-invariant metric (which can be readily constructed if G is compact), this action is gauge-invariant. $|F_A|^2$ is sometimes called the **Yang-Mills density**.

Proposition 1.1. *The Euler-Lagrange equations of this action are*

$$d_A^* F_A = 0. \tag{1.2}$$

Proof. This is an exercise in variational calculus. I'll skip most of the algebra. Observe that

$$\begin{aligned} F_{A+t\tau} &= d(A+t\tau) + (A+t\tau)\wedge(A+t\tau) \\ &= F_A + t d_A\tau + t^2\tau\wedge\tau. \end{aligned}$$

Then,

$$|F_{A+t\tau}|^2 = |F_A|^2 + 2t\langle d_A\tau, F_A\rangle + t^2(\cdots),$$

so

$$0 = \frac{d}{dt}S(A+t\tau) = 2\int_X \langle d_A\tau, F_A\rangle\, d\mu = 2\int_X \langle \tau, d_A^*F_A\rangle\, d\mu.$$

Hence the equations of motion are

$$d_A^* F_A = 0.$$

□

An **instanton** is a topologically nontrivial solution to the classical equations of motion with finite action.

Proposition 1.2. *Anti-self-dual connections are instantons, i.e., topologically nontrivial solutions to (1.2).*

Proof. First we show that an ASD connection solves the equations of motion. The main fact is

$$\star d_A^* F_A = d_A \star F_A,$$

so if A is ASD, $d_A \star F_A = -d_A F_A = 0$ by the Bianchi identity.

When $\dim X = 4$ and $G = SU(n)$, ASD connections are topologically nontrivial: for skew-adjoint matrices ($A^* = -A$, where $*$ is conjugate transpose), i.e., $\mathfrak{u}(n)$,

$$\operatorname{tr}(\xi \wedge \xi) = -|\xi|^2,$$

so

$$\operatorname{tr}\left(F_A^2\right) = -\left(\left|F_A^+\right|^2 - \left|F_A^-\right|^2\right) d\mu.$$

$|F_A|^2 = F_A \wedge \star F_A = \left|F_A^+\right|^2 + \left|F_A^-\right|^2$, so a connection is ASD if and only if $\operatorname{tr}\left(F_A^2\right) = |F_A|^2 \, d\mu$ at every $x \in X$. Recall that for $SU(n)$ bundles, $c_1(E)$ vanishes because $\operatorname{tr}(F_A) = 0$, so

$$c_2(E) = \frac{1}{8\pi^2} \int_X \operatorname{tr}\left(F_A^2\right) d\mu.$$

Hence,

$$S(A) = \int_X |F_A|^2 \, d\mu = \int_X \left|F_A^-\right|^2 d\mu + \int_X \left|F_A^+\right|^2 d\mu \geq 8\pi^2 c_2(E),$$

with the bound achieved precisely when A is ASD. For this reason, physicists often refer to $c_2(E)$ as the "**instanton number**." ☐

2. ADHM Construction

Let

$$F_{ij} := [\nabla_i, \nabla_j] = \frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i + [A_i, A_j].$$

Then, instanton equation $F_A^+ = 0$ becomes

$$\begin{aligned} F_{12} + F_{34} &= 0, \\ F_{14} + F_{23} &= 0, \\ F_{13} + F_{42} &= 0. \end{aligned} \tag{2.1}$$

The ADHM (Atiyah, Drinfeld, Hitchin, and Manin) construction gives a way of producing ASD connections from linear algebraic data. The idea is to take a "Fourier transform" of the ASD equations to produce a set of matrix equations which can be more readily solved.

Substituting $D_1 := \nabla_1 + i\nabla_2$, $D_2 := \nabla_3 + i\nabla_4$, the equations (2.1) become

$$[D_1, D_2] = (F_{13} + F_{42}) + i(F_{23} + F_{14}) = 0,$$

$$[D_1, D_1^*] + [D_2, D_2^*] = -2i(F_{12} + F_{34}) = 0,$$

so we can reduce the ASD equations to these "complex" covariant derivatives.

Because the ASD equation and $|F_A|^2$ are conformally invariant, an ASD connection on $\mathbb{R}^4$ with $S(A) < \infty$ can be regarded as an ASD connection on S^4.

2.1. **ADHM Data.** Let $U \cong \mathbb{C}^2$ as a complex manifold, with coordinates (z_1, z_2). The inputs for the ADHM construction consist of:

(1) A k-dimensional complex vector space H with a Hermitian metric.
(2) An n-dimensional complex vector space E_∞, with Hermitian metric and symmetry group $SU(n)$.
(3) A linear map $T \in V^* \otimes \hom(H, H)$ defining maps $\tau_1, \tau_2 : H \to H$.
(4) Linear maps $\pi : H \to E_\infty$ and $\sigma : E_\infty \to H$.

The **ADHM equations** are

$$\begin{aligned} [\tau_1, \tau_2] + \sigma\pi &= 0, \\ [\tau_1, \tau_1^*] + [\tau_2, \tau_2^*] + \sigma\sigma^* - \pi^*\pi &= 0. \end{aligned} \tag{2.2}$$

If $\tau_1, \tau_2, \sigma, \pi$ satisfy these equations, then the maps

$$\alpha := \begin{bmatrix} \tau_1 \\ \tau_2 \\ \pi \end{bmatrix}, \qquad \beta := \begin{bmatrix} -\tau_2 & \tau_1 & \sigma \end{bmatrix}$$

define a complex

$$H \xrightarrow{\alpha} H \otimes U \oplus E_\infty \xrightarrow{\beta} H$$

because

$$\beta\alpha = [\tau_1, \tau_2] + \sigma\pi = 0.$$

In fact, it defines a whole $\mathbb{C}^2$-family of complexes because we can replace (τ_1, τ_2) by $(\tau_1 - z_1 \cdot 1, \tau_2 - z_2 \cdot 1)$ for any point $(z_1, z_2) =: x \in U$. We can then define a family of maps

$$R_x : H \otimes U \oplus E_\infty \to H \oplus H$$

$$R_x := \alpha_x^* \oplus \beta_x$$

and, if α_x is injective and β_x is surjective, then R_x is surjective and $\ker R_x = (\operatorname{im} \alpha_x)^\perp \cap \ker \beta_x$.

Definition. A collection $(\tau_1, \tau_2, \sigma, \pi, E_\infty, H)$ of ADHM data is an **ADHM system** if

(1) it satisfies the ADHM equations (2.2), and
(2) the map R_x is surjective for each $x \in U$.

2.2. **ADHM Construction.** How can we use this information to construct a connection? Suppose that we have an ADHM system. Now, construct the vector bundle $E \to U$ with fibers

$$E_x = \ker R_x = \ker \beta_x / \operatorname{im} \alpha_x.$$

(E_x is the **cohomology bundle** of α,β).

Proposition 2.1. *There is a holomorphic structure $\mathscr{E}$ on E.*

(Proof omitted in the interest of time).

Let $i : E_x \hookrightarrow H \otimes U \oplus E_\infty$ be inclusion, $P_x^\alpha : H \otimes U \oplus E_\infty \to (\operatorname{im} \alpha_x)^\perp$ and $P_x^\beta : H \otimes U \oplus E_\infty \to \ker \beta_x$ denote orthogonal projections, and $P_x := P_\alpha \circ P_\beta = P_\beta \circ P_\alpha$

be projection onto E_x. $H \otimes U \oplus E_\infty$ comes equipped with a flat product connection d, and we can define an induced connection A on E by, for a section $s : U \to E$,

$$d_A s := Pdi\,(s)\,.$$

By virtue of this construction, A is unitary and compatible with the holomorphic structure on E. It is also ASD.

Theorem 2.2 (ADHM)**.** *The assignment $(\tau_1, \tau_2, \sigma, \pi) \to d_A = Pdi$ sets up a one-to-one correspondence between*

(1) *equivalence classes of ADHM data for group $SU(n)$ and index k, and*
(2) *gauge equivalence classes of finite energy ASD $SU(n)$-connections A over $\mathbb{R}^4$ with $c_2(A) = k$.*

Note that $(g, h) \in U(k) \times SU(n)$ acts on ADHM data by

$$(\tau_1, \tau_2, \sigma, \pi) \mapsto \left(g\tau_1 g^{-1}, g\tau_2 g^{-1}, g\sigma h^{-1}, h\pi g^{-1}\right),$$

so we mean classes of ADHM data up to this equivalence.

2.3. **Example: BPST Instanton.** The simplest example is to take $k = 1$ and $n = 2$. This corresponds to solutions on $SU(2)$ bundles with $c_2 = 1$. Then, τ_1, τ_2 are just complex numbers, σ and π are complex vectors, and the ADHM equations become

$$\sigma \cdot \pi = 0, \qquad |\sigma|^2 = |\pi|^2\,.$$

Pick $\pi = (1, 0)$ and $\sigma = (0, 1)$, then for (τ_1, τ_2), have

$$\alpha_x^* = \begin{bmatrix} \tau_1 \\ \tau_2 \\ 1 \\ 0 \end{bmatrix}^* = \begin{bmatrix} \overline{\tau_1} & \overline{\tau_2} & 1 & 0 \end{bmatrix}, \qquad \beta_x = \begin{bmatrix} -\tau_2 & \tau_1 & 0 & 1 \end{bmatrix},$$

and in general: replace (τ_1, τ_2) by $(\tau_1 - z_1 \cdot 1, \tau_2 - z_2 \cdot 1)$ for any point $(z_1, z_2) =: x \in U$, so

$$R_x = \begin{bmatrix} \overline{\tau_1} - \overline{z_1} & \overline{\tau_2} - \overline{z_2} & 1 & 0 \\ -\tau_2 + z_2 & \tau_1 - z_1 & 0 & 1 \end{bmatrix}.$$

In particular, for $(\tau_1, \tau_2) = (0, 0)$, have

$$R_x = \begin{bmatrix} -\overline{z_1} & -\overline{z_2} & 1 & 0 \\ z_2 & -z_1 & 0 & 1 \end{bmatrix}.$$

A unitary basis for R_x is

$$\{\sigma_1, \sigma_2\} = \left\{ \frac{1}{1 + |x|^2} \begin{bmatrix} 1 \\ 0 \\ \overline{z_1} \\ -z_2 \end{bmatrix}, \frac{1}{1 + |x|^2} \begin{bmatrix} 0 \\ 1 \\ \overline{z_2} \\ z_1 \end{bmatrix} \right\}.$$

Suppose that in this trivialization we let $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, and the connection matrix is

$$A = \sum A_i dx_i,$$

so A_i is the matrix with (p, q)th entry

$$\langle \nabla_i \sigma_p, \sigma_q \rangle = \left\langle \frac{\partial \sigma_p}{\partial x_i}, \sigma_q \right\rangle.$$

Then, written out in full, the connection form is

$$A = \frac{1}{1+|x|^2}\left(\theta_1 \mathbf{i} + \theta_2 \mathbf{j} + \theta_3 \mathbf{k}\right),$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are a standard basis for $\mathfrak{su}(2)$ and

$$\begin{aligned} \theta_1 &= x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3 \\ \theta_2 &= x_1 dx_3 - x_3 dx_1 - x_4 dx_2 + x_2 dx_4, \\ \theta_3 &= x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2 \end{aligned}$$

is such that $d\theta_1, d\theta_2, d\theta_3$ is a basis for the ASD two-forms on $\mathbb{R}^4$. The curvature $F_A = dA + A \wedge A$ is then

$$F_A = \left(\frac{1}{1+|x|^2}\right)^2 \left(d\theta_1 \mathbf{i} + d\theta_2 \mathbf{j} + d\theta_3 \mathbf{k}\right),$$

and we can recover the other degrees of freedom lost in our choices of π, σ, τ_1, τ_2 by translations $x \mapsto x - y$ and dilations $x \mapsto x/\lambda$ to obtain other connections with

$$\left|F_{A(y,\lambda)}\right| = \frac{\lambda^2}{\left(\lambda^2 + |x-y|^2\right)^2}.$$

3. Moduli Space of ASD Connections

Definition 3.1. Let $E \to X$ be a bundle over a compact, oriented Riemannian 4-manifold X. The **moduli space of ASD connections** M_E is the set of gauge equivalence classes of ASD connections on E.

Recall that a **gauge transformation** is an automorphism $u : E \to E$ respecting the structure on the fibers and reducing to the identity map on X. It acts on a connection by the rule

$$\nabla_{u(A)} s = u \nabla_A \left(u^{-1} s\right) = \nabla_A s - \left(\nabla_A u\right) u^{-1} s,$$

where the covariant derivative $\nabla_A u$ is formed by regarding it as a section of the vector bundle $End(E)$. In local coordinates, this looks like

$$u(A) = uAu^{-1} - (du)\, u^{-1}.$$

The curvature transforms as a tensor under gauge transformations:

$$F_{u(A)} = uF_A u^{-1}.$$

For connections on principal bundles $P \to X$, this has a somewhat nicer expression: If $u : P \to P$ satisfies

(1) $u(p \cdot g) = u(p) \cdot g$ and
(2) $\pi(u(p)) = \pi(p)$

for all $g \in G$, and $A \in \Omega^1_P(\mathfrak{g})$ is a connection,

$$u(A) := \left(u^{-1}\right)^* A.$$

Now we turn to some results about the structure of this moduli space.

3.1. **Uhlenbeck's Theorems.** First, there are a few technical results due to Uhlenbeck that allow us to leverage tools from the study of elliptic differential equations to make statements about ASD connections.

Theorem 3.2 (Uhlenbeck)**.** *There are constants ϵ_1, $M > 0$ such that any connection A on the trivial bundle over $\overline{B}^4$ with $||F_A||_{L^2} < \epsilon_1$ is gauge equivalent to a connection $\tilde{A}$ over B^4 with*

(1) $d^*\tilde{A} = 0$,
(2) $\lim_{|x|\to 1} \tilde{A}_r = 0$, *and*
(3) $||\tilde{A}||_{L^2_1} \leq M||F_{\tilde{A}}||_{L^2}$.

Moreover for suitable constants ϵ_1, M, $\tilde{A}$ is uniquely determined by these properties, up to $\tilde{A} \mapsto u_0\tilde{A}u_0^{-1}$ for a constant u_0 in $U(n)$.

First, some notes about the theorem:

$$||\tilde{A}||^2_{L^2_1} = \int_{B^4} \left|\nabla\tilde{A}\right|^2 + \left|\tilde{A}\right|^2 d\mu$$

is the Sobolev norm. $d^*\tilde{A}$ is the "Coulomb" gauge condition (the importance of which will be explained in the following section). Finally, $\lim_{|x|\to 1} \tilde{A}_r = 0$ means that, for $\tilde{A}_r(\rho,\sigma)$ a function on S^3, this function tends to 0 as $r \to 1$.

The main power of Uhlenbeck's Theorem is that it turns a system of nonlinear, nonelliptic differential equations into an elliptic one. This section provides a sketch of why that might be a desirable thing to do. Recall the d^+ operator, defined by

$$d^+ = \left(\frac{1}{2}(1+*)\right) \circ d,$$

which maps

$$d^+ : \Omega^1_X \to \Omega^+_X.$$

The ASD equation $F_A^+ = 0$ then becomes, in local coordinates,

$$d^+A + (A \wedge A)^+ = 0. \tag{3.1}$$

This is a nonlinear, non-elliptic equation.

When $d^*\tilde{A} = 0$,

$$d^* + d : \oplus_i\Omega_X^{2i+1} \to \oplus_i\Omega_X^{2i}$$

is elliptic, so if $H^1(X) = 0$, then all 1-forms are othogonal to $\ker(d + d^*)$.

Elliptic differential operator theory implies that

$$||A||_{L^2_k} \leq \text{const.}\left(||d^*A||_{L^2_{k-1}} + ||dA||_{L^2_{k-1}}\right) \tag{3.2}$$

for all k. When $d^*A = 0$, this becomes

$$||A||_{L^2_k} \leq \text{const.} \cdot ||F_A||_{L^2_{k-1}},$$

and the ASD equation can be replaced by the elliptic differential equation

$$\delta A = 0,$$

where $\delta = d^+ + d^*$ is an elliptic operator.

The main consequence of Uhlenbeck's Theorem relevant to the discussion of ASD connections comes from combining it with the following theorem:

Theorem 3.3 (Uhlenbeck)**.** *There exists a constant* $\epsilon_2 > 0$ *such that if* $\tilde{A}$ *is any ASD connection on the trivial bundle over* B^4 *which satisfies* $d^*\tilde{A} = 0$ *and* $||\tilde{A}||_{L^4} \leq \epsilon_2$*, then for all interior domains* $D \subset B^4$ *and* $l \geq 1$*,*

$$||\tilde{A}||_{L^2_l(D)} \leq M_{l,D}||F_{\tilde{A}}||_{L^2(B^4)}$$

for a constant $M_{l,D}$ *depending only on* l *and* D*.*

Combining this with Theorem 3.2 gives

Corollary 3.4. *For any sequence of ASD connections* A_α *over* $\overline{B}^4$ *with* $||F(A_\alpha)||_{L^2} \leq \epsilon$*, there is a subsequence* α' *and gauge equivalent connections* $\tilde{A}_{\alpha'}$ *which converge in* C^∞ *on the open ball.*

3.2. **Results about the Moduli Space.** Putting our previous results together, we get the following statements:

Theorem 3.5 (Uhlenbeck's Removable Singularities)**.** *Let* A *be a unitary connection over the punctured ball* $B^4\backslash\{0\}$ *which is ASD with respect to a smooth metric on* B^4*. If*

$$\int_{B^4\backslash\{0\}} |F_A|^2 < \infty,$$

then there is a smooth ASD connection over B^4 *gauge equivalent to* A *over the punctured ball.*

Note that this theorem implies that, for example, the ADHM construction gives all of the ASD connections on S^4 (not just $\mathbb{R}^4$).

Let $M_k(G)$ denote the moduli space of ASD connections up to gauge transformation with $c_2 = k$, and $\overline{M_k}(G)$ denote the closure of $M_k(G)$ in the space of "ideal connections." An **ideal connection** is a connection with curvature densities possibly having δ-measure concentrations at up to k points of X, i.e., of the form

$$|F_A|^2 + 8\pi^2 \sum_{i=1}^{n} \delta_{x_i}$$

Then,

Theorem 3.6. *Any infinite sequence in* M_k *has a weakly convergent subsequence in* $\overline{M_k}$*, with limit point in* $\overline{M_k}$*.*

Corollary 3.7. *The space* $\overline{M_k}$ *is compact.*

What do these spaces look like locally? Let $\mathscr{G}$ denote the group of gauge transformations of $E \to X$, and

$$\Gamma_A = \{u \in \mathscr{G} : u(A) = A\},$$

the isotropy group of A. Then,

Proposition 3.8. *If* A *is an ASD connection over* X*, a neighborhood of* $[A]$ *in* M *is modeled on a quotient* $f^{-1}(0)/\Gamma_A$*, where*

$$f : \ker \delta_A \to \operatorname{coker} d_A^+$$

is a Γ_A*-equivariant map and* $\delta_A = d_A^* + d_A^+$*.*

References

[ADHM] M. Atiyah, V. Drinfeld, N. Hitchin, and Y. Manin, “Construction of Instantons,” Physics Letters A **65** (3) 185–187.

[DK] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Univ. Press, 1991.

[L] A. J. Lindenhovius, “Instantons and the ADHM Construction,” URL: http://dare.uva.nl/cgi/arno/show.cgi?fid=333494