Gauge Theory Masterclass

Heegaard Floer 1

Goals:

1) Basic definitions
2) 3d geometric content (using sutured manifolds)
3) Recent extension: bordered HF, & how to use it to compute.

Morse Theory

Conventions: manifolds will be compact, smooth, & if necessary oriented. Let $f : M \longrightarrow \mathbb{R}$.

Definition:

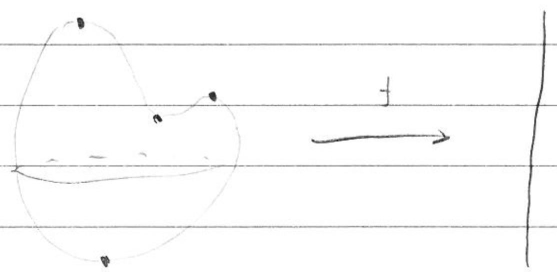
A critical point of f is a point $p \in M$ s.t. $df(p) = 0$. $Crit(f)$ will denote the set of such. A critical value is its image.

f is Morse if $\forall p \in Crit(f)$, the 2nd derivative tells you the convexity of f. i.e.

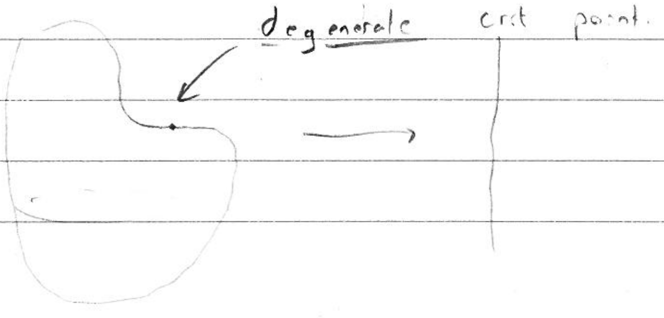
$$Hess_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

Example

Height on a sphere



4 critical points.



Morse lemma:

If $p \in Crit(f)$ is non-degenerate, then there are coords $x_1, \ldots, x_n$ near p such that

$$f(x_1, \ldots, x_n) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 .$$

Moreover i is well-defined: the index of p.

We might call such co-ordinates "nice".

We can study the topology of M by looking at level sets: their topology changes precisely at the critical points.

Proposition

Let f be Morse. Suppose $f^{-1}([t_1, t_2])$ has no critical points ($[t_1, t_2]$ has no critical values). Then

$$f^{-1}(-\infty, t_1] \cong f^{-1}(-\infty, t_2].$$

Proof:

Let V be a vector field on M s.t. if $p \notin Crit(f)$, $V_p(f) > 0$ ("gradient-like"). Scale V s.t. $V_p(f) = 1$ if $p \in [t_1, t_2]$. Then time 1 flow of V is the desired diffeo. □

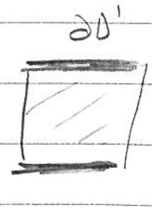
Definition:

An n-dim^l k-handle is $D^k \times D^{n-k}$ attached along $(\partial D^k) \times D^{n-k}$

e.g. $n=2$, $k=0$. $D^0 \times D^2$ attached along $\emptyset$

$n=2$ $k=1$ $D^1 \times D^1$ attached along two sides:

$n=2$ $k=2$ $D^2 \times D^0$ attached along ∂D^2.



Proposition

Suppose $\exists!\ p \in Crit(f)$ such that $t_1 < f(p) < t_2$.
Then $f^{-1}(-\infty, t_2]$ is obtained from $f^{-1}(-\infty, t_1]$
by attaching an n-dim ind(p) - handle.

Theorem:

For any (smooth) M, there exists a Morse function on M.
Moreover f can be chosen so that if $p, q \in Crit(f)$
& $f(p) \leq f(q)$, then $ind(p) \leq ind(q)$

CF Milnor - Morse Theory
Lectures on h-cobordism Theorem.

In fact, we can make f self-indexing, i.e. $f(p) = ind(p)$.

3-dimensions:

Definition:

A 3d handlebody is

1) The regular neighbourhood of a connected graph in $\mathbb{R}^3$

boundary sum, → 2) $(S^1 \times D^2) \# (S^1 \times D^2) \# \cdots \# (S^1 \times D^2)$

not connected sum 3) A connected 3-manifold built entirely from 0 & 1 handles

4) interior of Σ_g

e.g Not $S^3 \setminus nbd($ $)$

Proposition

Any 3-manifold can be split as

$$Y = H_1 \cup_\phi H_2$$

H_i handlebodies of genus g

$\phi : \partial H_1 \xrightarrow{\sim} \partial H_2$

Proof:

Let $f : Y \to \mathbb{R}$ be a self-indexing Morse function. Choose $1 < t < 2$. So $f^{-1}(-\infty, t]$ is a handlebody (condition 3). Also $f^{-1}([t, \infty)) = (-f)^{-1}(-\infty, -t]$ is a handlebody, since $\operatorname{ind}_f(p) = n - \operatorname{ind}_{-f}(p)$. □

Corollary

Any 3-manifold Y can be built by

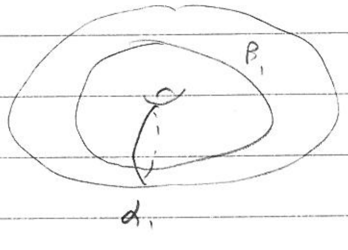
1) Start with Σ_g, closed orientable genus g surface.
2) Thicken to $\Sigma \times [-\varepsilon, \varepsilon]$
3) Attach 3d 2-handles along curves $\alpha_1, \dots, \alpha_g \subseteq \Sigma \times -\varepsilon$
   $\beta_1, \dots, \beta_g \subseteq \Sigma \times \varepsilon$

   so that $\partial(\text{result})$ is $S^2 \sqcup S^2$
4) Fill both S^2s with B^3s.

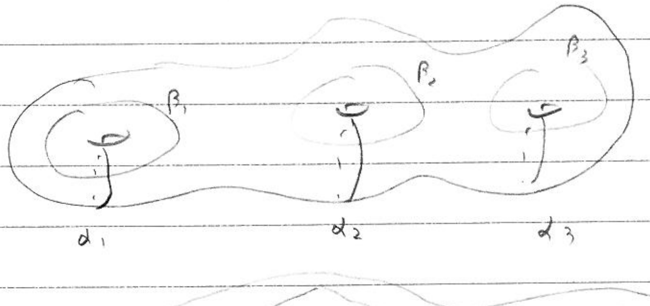
So Y is determined by Σ_g, $\{\alpha_1, \dots, \alpha_g\}$, $\{\beta_1, \dots, \beta_g\}$ with $\alpha_i \cap \alpha_j = \emptyset = \beta_i \cap \beta_j$ if $i \neq j$.

This is called a <u>Heegaard diagram</u>.

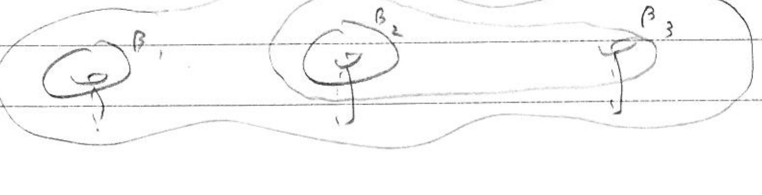
Example



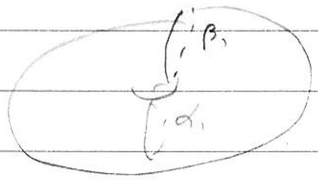
yields S^3



yields S^3 also.



as does this.



yields $S^2 \times S^1$.

<u>Exercise</u> produce a genus 1 Heegaard diagram for $\mathbb{RP}^3$.

Morse Homology

Let $f : M \longrightarrow \mathbb{R}$ be Morse.

<u>Theorem</u> (Morse)

There's a chain complex $C_*(f)$ such that

1) $C_m(f) = \mathbb{Z}\langle$ index m critical points of $f \rangle$
2) $H_*(f) \cong H_*^{sing}(M)$.

<u>Proof</u>:

f yields a handle decomposition, hence a cell decomposition. The cellular chain complex has the desired property. □

<u>Corollary</u>:

If $f : \Sigma_g \longrightarrow \mathbb{R}$ is Morse, then f has at least $2g+2$ critical points.

A vector field V on M is called <u>gradient - like</u> if

1) $V_p(f) > 0 \quad \forall\, p \notin Crit(f)$
2) In nice coords near critical p,

$$V(x_1, \ldots, x_n) = -2x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} - \cdots - 2x_i \frac{\partial}{\partial x_i} + 2x_{i+1} \frac{\partial}{\partial x_{i+1}} + \cdots + 2x_n \frac{\partial}{\partial x_n}.$$

(i.e. looks like $grad(f)$).

Given such, we can consider gradient flow lines $\gamma : \mathbb{R} \to M$ s.t $\gamma'(t) = V(\gamma(t))$. They converge to critical points as $t \longrightarrow \pm\infty$

For $p \in Crit(f)$, consider

$$U(p) := \{ x \in M : \lim_{t \to -\infty} \gamma_x(t) = p \}$$

$$D(p) := \{ x \in M : \lim_{t \to \infty} \gamma_x(t) = p \}$$

where γ_x is the unique flow line through x.

Lemma:

These are discs.

Definition

The pair (f, v) is called Morse-Smale if $\forall p, q \in Crit(f)$, $U(p) \pitchfork D(q)$.

Then $U(p) \cap D(q)$ is a manifold of dimension $ind(q) - ind(p)$.
$\mathbb{R} \curvearrowright M(p,q) = U(p) \cap D(q)$ by translation along the flow lines.

Proposition:

The differential on $C_*(f)$ is given by

$$\partial(p) = \sum_{\substack{q \\ ind(q) = ind(p) - 1}} |M(p,q)/\mathbb{R}| \, q$$

Perutz 1

Gauge Theory & Symplectic Geometry

1. What gauge theory has done for us.

4-manifolds:

Let X be a 4-manifold: always closed, connected, oriented & smooth.

The key differential geometric feature: the bundle Ω^2 of 2-forms has an involution: Hodge $*$.

So we fix a conformal class $\wedge$ of Riemannian metrics.

Ω^2 splits as $\Lambda^+ \oplus \Lambda^-$: ± 1 eigenspaces for $*$: self-dual & anti-self-dual 2-forms.

Key algebro-topological feature: the lattice $H = H^2(X;\mathbb{Z})/\text{torsion}$ carries a unimodular symmetric bilinear form Q_X : the intersection form :

$$Q_X(a,b) = (a \cup b)[X].$$

Uni-modular is Poincaré duality.

Realizations of Q_X:

By Poincaré duality $H \cong H_2(X;\mathbb{Z})/\text{torsion}$. In H_2, Q_X counts signed intersections of embedded oriented surfaces in general position.

Also, consider $H \otimes \mathbb{R} \xrightarrow[\text{de Rham}]{\sim} H^2_{dR}(X)$. In these terms

$$Q_X(\alpha,\beta) = \int_X \alpha \wedge \beta .$$

Hodge Theorem

$H^2_{dR}(X) \cong \mathcal{H}^2_g$: harmonic 2-forms wrt metric g.

But $\mathcal{H}^2_g = \mathcal{H}^+_g \oplus \mathcal{H}^-_g$, where

$\mathcal{H}^\pm_g = \Omega^\pm \cap \mathcal{H}_g$. $(\Omega^\pm = \Gamma(\Lambda^\pm))$.

The splitting descends because the Laplacian is built from $*$.

Furthermore, on $\mathcal{H}^+$,

$$Q_X(\alpha,\alpha) = \int_X \alpha\wedge\alpha = \int_X \alpha \wedge *\alpha = \int |\alpha|^2 \, d\text{vol} > 0$$

So Q_X is >0 on $\mathcal{H}^+$, <0 on $\mathcal{H}^-$:
maximal positive / negative subspaces

Definition

$b^{\pm} = \dim \mathcal{H}^{\pm}$ (note: doesn't need Hodge theory)

$\sigma(X) = \sigma(Q_X) = b^+ - b^-$ signature of X.

Fact:

$\sigma(X)$ is invariant under oriented cobordism.

Examples

$\sigma(\mathbb{C}P^2) = 1$

so $\sigma(\#^n \mathbb{C}P^2 \# \#^m \overline{\mathbb{C}P^2}) = n - m$.

Instantons:

Let $\begin{smallmatrix} P \\ \downarrow \\ X \end{smallmatrix}$ be a principal Lie bundle, for G some Lie group. We'll talk about connections A on P.

So curvature $F_A \in \Omega^2(X, \text{ad } P)$

It splits as $F_A = F_A^+ + F_A^-$ into SD, ASD parts.

Definition

An *instanton* on P is a connection A such that $F_A^+ = 0$. (weaker than flat).

Definition

The group of *gauge transformations* is $\mathcal{G} = \text{Aut}(P)$
$= \Gamma(\text{Ad } P)$.

This acts on connections: $u \in \mathcal{G}$. $u \cdot A = u^* A$
$= A - u^{-1} d_A u$.

Note $F_{u \cdot A} = u^{-1} F_A u$
$F^{\pm}_{u \cdot A} = u^{-1} F^{\pm}_A u$, so $\mathcal{G}$ preserves instantons.

Let $\mathcal{M}_X = \mathcal{M}_X(P, g)$ denote $\{\text{instantons}\}/\mathcal{G}$.

One can essentially find a section for the $\mathcal{G}$ action (up to covariant constant gauge transformations) by imposing <u>Coulomb gauge</u>

Fix some A_0. So $A - A_0 \in \Omega^1(\text{ad} P)$.

require $d^*_{A_0}(A - A_0) = 0$ "divergence zero".

This only works <u>locally</u>.

<u>Linearization</u>

linearization at A_0 of ASD & Coulomb gauge is

$$\delta_A := d^+_{A_0} + d^*_{A_0} : \Omega^1 \to \Omega^+ \oplus \Omega^0.$$

This is <u>elliptic</u> as a differential operator, so standard theory tells us δ_A is Fredholm.

Now, if one can arrange for δ_A to be <u>surjective</u> (M_X cut out transversely). Then Fredholm implies $\mathcal{M}_X$ is locally homeomorphic to

$$\ker \delta_A / \text{Stab}_{\mathcal{G}} A$$

(very small)

Furthermore, now $\ker \delta_A = \text{Ind}\, \delta_A$, which can be computed by Atiyah - Singer.

<u>e.g.</u>: For $G = SU(2)$

$$\text{ind}\, \delta_A = 3(-1 + b_1 - b^+) + 8\, c_2(P)[X]$$

(3 = dim $SU(2)$)

Thus, if δ_A is onto, we get a moduli space of instantons, which is an orbifold.

<u>Note</u>: $\text{Stab}_{\mathcal{G}} A \subseteq G/Z(G)$ is $C_G(\text{Hol}(A))/Z(G)$

Examples

$G = U(1)$. So $\Omega^2_X(\text{ad} P) = \Omega^2_X(i\mathbb{R})$.

Say $Q_X < 0$: negative definite. e.g. $\#^n \overline{\mathbb{CP}}^2$.

Then $\mathcal{H}^2_g = \mathcal{H}^-_g$.

Chern-Weil says $[F_{A_0}] = -2\pi i \, c_1(P)$.

Given $A = A_0 + i\xi$, $F_A = F_{A_0} + i d\xi$, so we can make the curvature __any__ __rep^n__ of this cohomology class, e.g in $\mathcal{H}^-_g$, i.e an instanton.

If $b_1(X) = 0$, this A is unique mod $\mathcal{G}$ ← moduli is a point.

__abelian instantons.__

Donaldson's Diagonalisability Theorem

Assume $Q_X < 0$, and $\pi_1(X) = 1$.

Then there's a basis for $H^2(X; \mathbb{Z})$ in which Q_X has matrix $-\text{Id}$.

__Proof sketch:__

Fix a principal $SU(2)$-bundle $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$, with

$$c_2(P \times_{SU(2)} \mathbb{C}^2)[X] = 1.$$

Look at $\mathcal{M}_X(P, g)$.

First, abelian instantons: let

$$N_X = \{ \pm c \in H^2/\pm \text{Id} : Q_X(c, c) = -1 \}.$$

For any $c \in H^2$, $\exists !$ line bundle L s.t $c_1(L) = c$.

So if $\pm c \in N_X$, $P \times_{SU(2)} \mathbb{C}^2 \cong L \oplus L^*$.

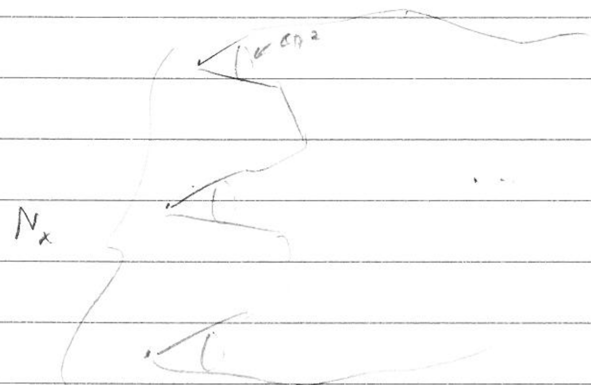
L carries a unique (up to $U(1)$-gauge) abelian instanton A_L. So put $B = A_L + A_{L^*}$ an instanton on P.

These are <u>reducible</u>: have non-trivial stabiliser. (it's $U(1)$)
Thus, we get a discrete set of instantons indexed by N_X.

Local model for M_X at an abelian reducible instanton is $\mathbb{C}^3/U(1)$, i.e. cone on $\mathbb{CP}^2$.

So get singularities corresp to N_X, whose link is $\mathbb{CP}^2$.

<u>Fact</u>: other instantons are irreducible (using π_1 trivial).
So local model, (if δ_A is onto) is $\mathbb{R}^5$



<u>Compactness</u>: If $[A] \in M_X$, Chern-Weil theory tells us

$$1 = \frac{1}{8\pi^2}\int \mathrm{Tr}\, F_A^2 = \frac{1}{8\pi^2}\int \|F_A\|^2 \, d\mathrm{vol}_g$$

So $\mu = \frac{1}{8\pi^2}\|F_A\|^2$ is a probability measure on X.

<u>Uhlenbeck</u> proved: Suppose $[A_n]$ is a sequence of instantons with no cvgnt subsequence. Then $\exists$ a subsequence, & points $x_1, \dots, x_m \in X$ such that

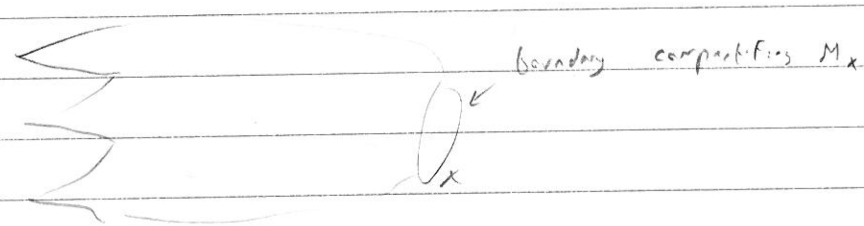
- $[A_n] \longrightarrow [A_\infty]$ on $X \setminus \{x_1, \dots, x_m\}$
- $\mu_n \longrightarrow \frac{1}{8\pi^2 m}\sum \delta_{x_i}$, sum of Dirac deltas.

Near each x_i, A_n "bubbles off" an instanton over S^4.

c_2 adds on the bubbling. Thus when $c_2 = 1$, only one bubble can occur.

Donaldson: $M_{conc} \subseteq M_X$: connections st 1-d of μ lies in an ε-ball.
Then $M_X \setminus M_{conc}$ is compact, & $M_{conc} \cong (0,1) \times X$.

So M looks like



So $\exists$ cobordism X to $\coprod_{N_X} \mathbb{CP}^2$

So $\underset{\underset{-b_2}{\parallel}}{\sigma(X)} = \sum_{|N_X|} \pm 1 \geq -|N_X|$

So $b_2 \leq |N_X|$

Lemma: (Do-it-yourself)
In a -ve definite unimodular lattice, if $\# \{ \pm c \mid Q_X(c,c) = -1 \}$ is $\geq$ rank, then $Q_X = -Id$. $\square$

Sutured Floer Homology: The case of Links

Knot invariants: Laurent polynomials associated to quantum groups.
poly $\rightarrow$ homology: categorification.

Heegaard Floer homology $\widehat{HFK}$ for a knot.
- It's bigraded — Alexander & Maslov.
- The Euler characteristic: alternating sum of Maslov grading gives coefficients of the Conway-Alexander polynomial

Alexander polynomial of $K \subseteq S^3$. Take the universal abelian cover of $S^3 \setminus K$: a $\mathbb{Z}$-cover, as $H_1(S^3 \setminus K)$.

Can then look at $H_1(\widetilde{S^3 \setminus K})$. The group of deck transformations acts on it, as does its group ring $\mathbb{Z}[t, t^{-1}]$.

The alexander polynomial $\Delta(t)$ is the minimal poly in the kernel of the action (well defined up to units).

Conway polynomial

Declare $\nabla(\bigcirc) = 1$

$$\nabla(L_+) - \nabla(L_-) = (t^{1/2} - t^{-1/2})\,\nabla(L_0)$$

This normalises the above: **Conway - Alexander polynomial**

More Facts on $\widehat{HFK}$

- Max grading gives the **knot genus**

Definition:

A **Seifert surface** for a knot is an oriented surface embedded in $\mathbb{R}^3$ whose boundary is the knot.

The genus of K is the minimal genus of Seifert surfaces for K.

Seifert surfaces always exist, & the genus of K is $0 \iff K$ is the unknot.

Theorem (Neuwirth 1960)

genus $(K) \geq$ degree of the A-C polynomial.

Theorem (Ozsváth–Szabó 2001)

genus(K) = maximal grading in $\widehat{HFK}$

In particular, $\widehat{HFK}$ detects the unknot.

- $\widehat{HFK}$ determines whether a knot fibres

Definition

A knot is fibred if $S^3 \setminus K$ is a surface bundle over S^1

$$\Sigma \longrightarrow S^3 \setminus K$$
$$\downarrow$$
$$S^1$$

Alternatively, Seifert surfaces sweep out the complement.

Theorem (Neuwirth 1960)

K fibred ⟹ Alexander–Conway poly is monic

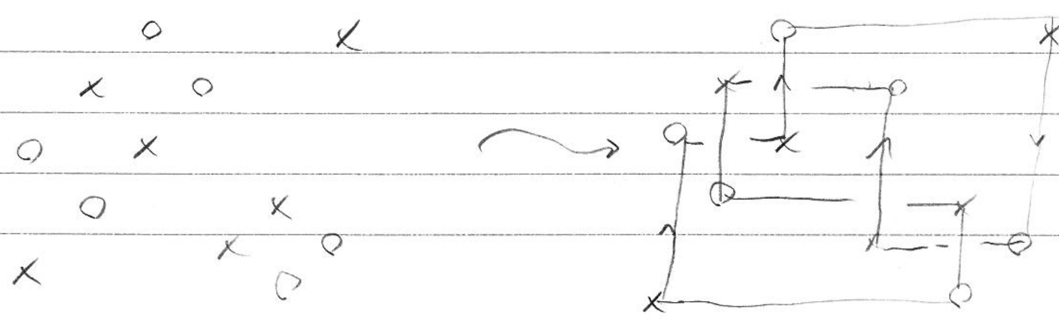
Theorem (Ghiggini–Ni 2006)

K fibred ⟺ in maximal Alexander grading, there's only a single entry.

$\widehat{HFK}$ is defined via pseudo-holomorphic curves.
We'll give a simple algorithm to compute $\widehat{HFK}$.

Grid Diagrams

Square diagrams with one X & one O in each row & column. They represent knots.



They always exist, & are unchanged by cyclic rotations of rows / columns.

Computing Δ

Take our grid diagram & make a matrix with entries $t^{-\text{winding number}}$.

Then take its determinant. It is

$$\pm t^{*} (1-t)^{n-1} \Delta(K;t)$$

, where n is the size of the diagram.

Exercise: Show that this poly is invariant directly: e.g. under moving a strand.

Computing HFK

Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- It has $n!$ generators: matchings between horizontal & vertical grid circles.
- Boundary ∂ switches corners on empty rectangles. (wrt X's, O's or other points). Sum over all ways of doing this.
- $\partial^2 = 0$: each term has a mate.
  - For disjoint rectangles can swap in either order
  - For rectangles sharing a corner, its

- Gradings: It's easier to do relative gradings.

: gradings change by $[0, -1]$

: changes by $[1, 1]$

: changes by $[-1, -1]$.

Exercise:

Compute $\widehat{HFK}$ (). (Find a small grid diagram).

Theorem (Manolescu - Ozsváth - Sarkar)

$$H_*(\widetilde{CK}) \cong \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$.

$V^{\otimes n-1}$ is like the factor of $(1-t)^{n-1}$.

Problem: Interpret this algebro-geometrically / representation theoretically.

Thurston Norm:

M a closed oriented 3-manifold. For any class $\alpha \in H_2(M)$, want a measure of complexity:
e.g. minimal genus embedded representative. Call this $g(\alpha)$.

Modify this slightly:

$$\chi_+(\Sigma) = \begin{cases} \max(0, -\chi(\Sigma)) & \Sigma \text{ connected} \\ \sum_i \chi_+(\Sigma_i) & \Sigma = \coprod_i \Sigma_i \end{cases}$$

Then put

$$\chi_+(\alpha) = \min \{ \chi_+(\Sigma) : \Sigma \subseteq M \text{ embedded}, [\Sigma] = \alpha \}.$$

Theorem:

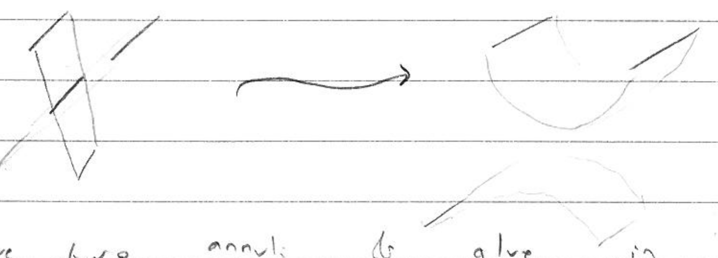
χ_+ is a pseudonorm on $H_2(M)$. (Tori have norm 0).

Proof sketch:

Subadditivity: $\chi_+(\alpha+\beta) \leq \chi_+(\alpha) + \chi_+(\beta)$ is the most interesting part.

Find Σ_1, Σ_2 representing α, β of minimal χ_+.
Look at $\Sigma_1 \cup \Sigma_2$. Might not be embedded.

If Σ_1, Σ_2 meet, wlog it is transversely, but one can do a surgery.



Remove two annuli & glue in two annuli, but this doesn't change χ, as $\chi(\text{annulus}) = 0$.

One then must fuss to deal with spherical components. $\square$

Heegaard Floer 2

Heegaard Floer Homology

Ozsváth - Szabó - Rasmussen

Formal structure:

{A closed smooth 4-manifold with $b_2^+ > 1$} $\longrightarrow$ Number.

Seiberg-Witten invariants. These have a TQFT (ish) structure, i.e. for a 3-manifold Y, closed, connected, oriented, we produce a "graded" abelian group $\widehat{HF}(Y)$

also "graded" $\mathbb{Z}[u]$-modules $HF^{\pm}(Y)$.

Then, given a cobordism W between 3-manifolds Y_1, Y_2, there is associated

$$\hat{F}_W : \widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2)$$

such that gluing cobordisms composes maps.

Given a closed 4-manifold, view it as a cobordism $\emptyset \rightarrow \emptyset$, but this is not the Seiberg-Witten invariant. It is 0 or something. Still, one can recover some 4-mfld invariants.

Now, given a knot $K \subseteq Y$, can produce a bigraded abelian group $\widehat{HFK}(Y, K)$.

$\widehat{HF}$ is <u>not</u> a generalised homology theory: not homology of a space.

<u>Technical Issues suppressed</u>

- Spin^c structures
- gradings
- orientations (we'll use $\mathbb{Z}/2\mathbb{Z}$-coeffs)

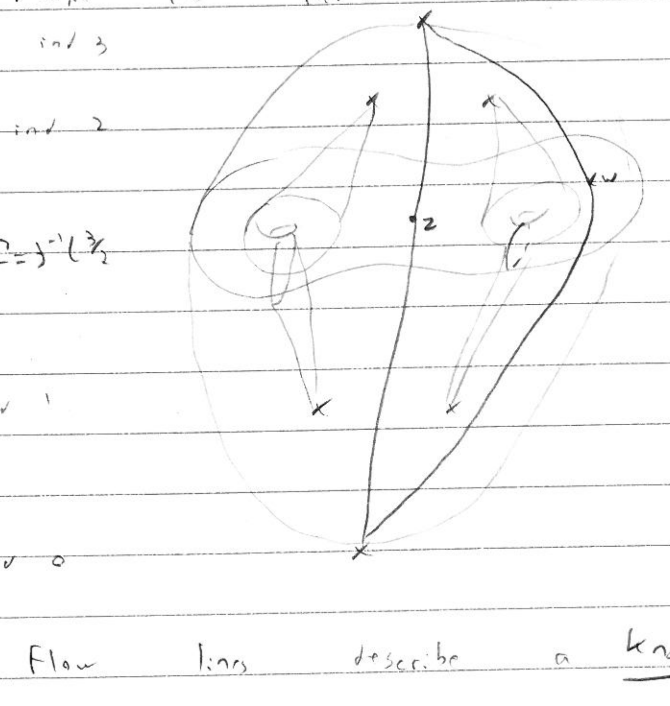
Defined in terms of Heegaard diagrams

$\mathcal{H} = (\Sigma_g, \underline{\alpha} = \{\alpha_1, \ldots, \alpha_g\}, \underline{\beta} = \{\beta_1, \ldots, \beta_g\})$

α's pairwise disjoint & LI in $H_1(\Sigma_g)$, similarly β's.

Also, fix a base point $z \in \Sigma \setminus (\underline{\alpha} \cup \underline{\beta})$. This specifies a flow line from the index 0 to index 3 critical point, hence roughly a ball $B^3 \subseteq Y$, our 3-manifold.

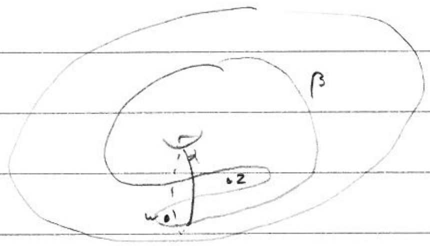
Fixing a second point $w \in \Sigma \setminus (\underline{\alpha} \cup \underline{\beta})$, get <u>two</u> flow lines from the index 0 to index 3 critical point



Two flow lines describe a <u>knot</u> $K \subseteq Y$.

In each handle body, the path $w \to z$ is the unique unknotted path missing the α or β discs.

Example



yields the trefoil

Can obtain any knot in this way, by choosing appropriate Heegaard splittings.

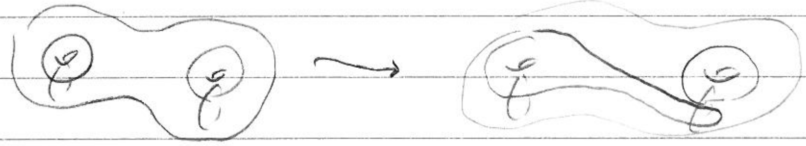
Theorem

If $\mathcal{H}$, $\mathcal{H}'$ are Heegaard diagrams representing the same Y or (Y, K), then you can get from $\mathcal{H}$ to $\mathcal{H}'$ by a sequence of

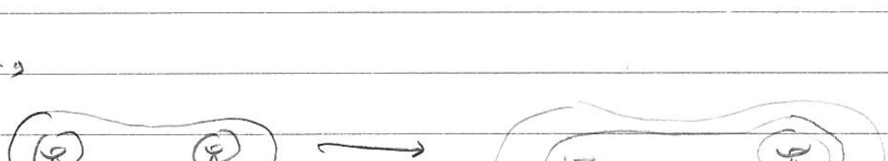
- isotopies
- handle slides
- stabilisations / destabilisations

not crossing z, w

Isotopy : e.g



Handle slide : e.g



Depends on a path connecting two circles: do a connect sum along it.

Stabilisation : connect sum with  .

Fix an $\mathcal{H}$. Want to think of $\underline{\alpha}$ as a single object. So take

$$\mathrm{Sym}^g \Sigma = \Sigma^g / S_g \quad \text{, where } S_g \text{ acts by re-ordering.}$$

<u>Facts:</u> 1) $\mathrm{Sym}^g \Sigma$ is a topological manifold.

2) A complex structure on Σ induces a smooth structure on $\mathrm{Sym}^g \Sigma$ (exercise).

3) A complex structure j_Σ induces a complex structure $\mathrm{Sym}^g(j_\Sigma)$ characterised by

$$(\Sigma^g, j_\Sigma^g) \longrightarrow (\mathrm{Sym}^g \Sigma, \mathrm{Sym}^g j_\Sigma) \text{ is holomorphic}$$

4) There are Kähler forms compatible with this structure.

$\alpha_1 \times \dots \times \alpha_g \in \Sigma^g$ maps to T_α diffeomorphically in $\mathrm{Sym}^g \Sigma$. Similarly T_β. These are <u>tori</u>.

The alg. intersection number $T_\alpha \cdot T_\beta = \begin{cases} |H_1(Y)| & \text{if finite} \\ 0 & \text{otherwise.} \end{cases}$

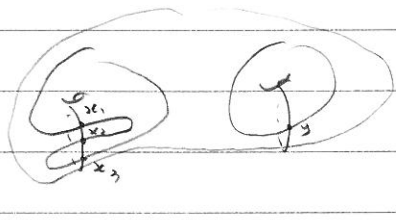
<u>Short definition</u>

$\widehat{HF}(Y) = HF(T_\alpha, T_\beta \subseteq \mathrm{Sym}^g(\Sigma \setminus z))$, Lagrangian Floer homology.

<u>Definition:</u>

$\widehat{CF}(Y) = \mathbb{Z}/2 \langle T_\alpha \cap T_\beta \rangle$

<u>Example:</u>



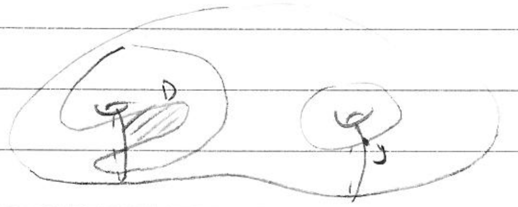
$\widehat{CF}(Y)$ generated by $\{x_1, y\}$, $\{x_2, y\}$ & $\{x_3, y\}$.

__Definition:__

A __Whitney disc__ from x to y in $T_\alpha \cap T_\beta$ is a continuous map $D^2 \longrightarrow \mathrm{Sym}^g \Sigma$ such that

$$-i \longmapsto x$$
$$+i \longmapsto y$$
$$\text{right arc} \longmapsto T_\alpha$$
$$\text{left arc} \longmapsto T_\beta$$

__Example__



$D \times \{y\} : D^2 \longrightarrow \Sigma \times \Sigma \longrightarrow \mathrm{Sym}^2 \Sigma$ is a whitney disc.

Let $\pi_2(x,y)$ denote the set of homotopy classes of whitney discs x to y.

A whitney disc u is __holomorphic__ if it entwines the complex structure:

$$du \cdot i = \mathrm{Sym}^2 j_\Sigma \cdot du.$$

Let $\mathcal{M}(\phi) = \{$ holomorphic whitney discs u in htpy class $\phi \in \pi_2(x,y) \}$.

__Proposition:__

$\mathcal{M}(\phi)$ is (generically) a finite dim^l^ manifold. It's dimension is given by algebro-topological data

$\mathrm{ind}(\phi) = \mu(\phi)$ (Maslov index).

Define a differential $\partial : \widehat{CF}(Y) \longrightarrow \widehat{CF}(Y)$ by

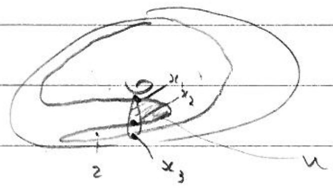
$$\partial(x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mathrm{ind}(\phi) = 1 \\ \phi \circ (z \times \mathrm{Sym}^{g-1}\Sigma) = 0}} \left| \mathcal{M}(\phi) / \mathbb{R} \right| \cdot y$$

$\mathbb{R}$ action on D fixing $\pm i$

Proposition : $\partial^2 = 0$

So put $\widehat{HF}(Y) = \ker(\partial) / \operatorname{Im}(\partial)$.
In fact there's a grading too...

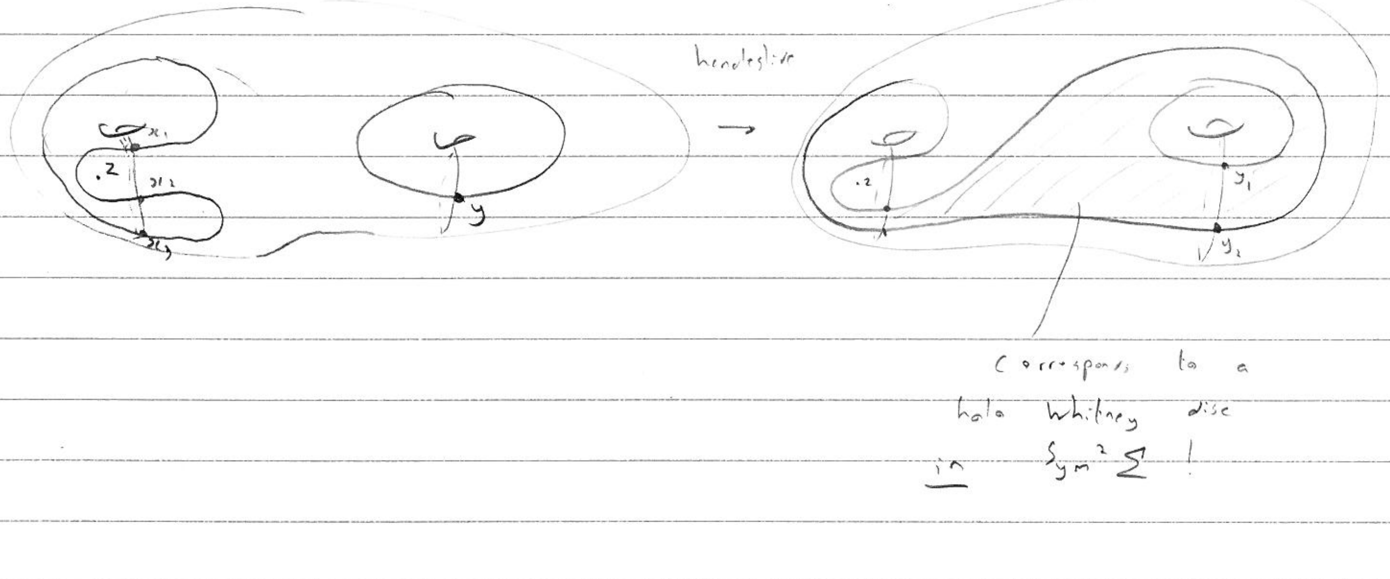
Examples :



Use the Riemann mapping theorem to make u holomorphic.
So $\partial x_1 = x_2$ (+ other terms ?)

Exercise There are no other homotopy classes of Whitney discs

So $\widehat{CF} = x_1 \xrightarrow{\partial} x_2 \quad x_3$
$\widehat{HF}(S^3) = \mathbb{Z}/2$.



Corresponds to a holo Whitney disc in $\mathrm{Sym}^2 \Sigma$!

Survey of Gauge Theory

- Instanton theory (Donaldson) Seiberg-Witten theory

$$\text{Seiberg-Witten theory} \cong \text{Heegaard Floer theory (Ozsváth-Szabó)}$$

With these, one can produce

1) Non-existence of smooth structures on 4d homotopy types (sharpest from homotopical refinements of SW).
2) Non-diffeomorphism of htpy equivalent 4-manifolds.
3) Restrictions on Riemannian geometry (e.g. Scalar curvature) for 4-manifolds (SW)
4) Diff topology of complex surfaces. (SW, D)
5) Symplectic 4-manifolds & contact 3-manifolds (SW, OS): e.g. · non-existence & inequivalence of symplectic / contact structures.
   · existence of pseudo-holo curves / periodic Reeb orbits.
6) Certification of minimal genus of surfaces in a fixed homology class. (D, SW, OS)
7) Uniqueness of surgery presentations of 3-manifolds (OS is strong) "property P" theorem (D)
8) Knots, & concordance between them (OS)

The Vortex Equations & SW Equations

Vortices:

Vortices are a 2d antecedent (dim' reduction) of 4d SW equations.

Fix Σ a closed smooth surface, g a Riemannian metric. This gives a conformal structure j, and a volume form α.

Further, take $\begin{matrix} L \\ \downarrow \\ \Sigma \end{matrix}$ a hermitian line bundle of degree $d = c_1(L)[\Sigma]$.

Finally, fix $\tau \in \mathbb{R}$.

<u>Fields</u>: A a unitary connection on L

ϕ a section of L.

The <u>vortex equations</u> $VOR(L, \tau)$ say

$$\bar{\partial}_A \phi = 0 \quad \in \Omega^{0,1}(\Sigma, L)$$

$$i F_A = (\tau - |\phi|^2)\,\alpha \quad \in \Omega^2(\Sigma).$$

There's a constraint on the <u>existence</u> of vortices from Chern-Weil theory.

$$d = \frac{1}{2\pi}\int_\Sigma i F_A = \frac{1}{2\pi}\int (\tau - |\phi|^2)\,\alpha$$

$$< \frac{1}{2\pi}\,\tau\ \text{area}(\Sigma).$$

So if $\exists$ a vortex with $\phi \neq 0$, you need $d < \frac{1}{2\pi}\tau\ \text{area}(\Sigma)$

There's a <u>gauge group</u> $\mathcal{G} = C^\infty(\Sigma, U(1))$, acting on $\{(A, \phi)\}$ via

$$u \cdot (A, \phi) = (u^* A,\ u\phi)$$

preserving vortices $VOR(L, \tau)$.

So one has moduli space $Vor(L, \tau) = \text{Solutions}/\mathcal{G}$.

It is a complex manifold of dimension d when $d < \frac{\tau}{2\pi}\text{area}(\Sigma)$. The complex structure I (integrable) is

$$I(a, \psi) = (*a,\ i\psi)$$

$$a \in i\Omega^1(\Sigma), \quad \psi \in \Gamma(L)$$

<u>How to think about them</u>

From now on, assume the C-W constraint.

First, $\bar\partial_A$ makes L into a <u>holomorphic</u> line bundle, so eq^n 1 says ϕ is a holomorphic section.

Have a map $\text{Vor}(L, \tau) \longrightarrow \{(\mathcal{L}, \phi) : \mathcal{L}$ holo line bundle, $\phi \neq 0$ holo section$\} / \mathcal{C}^\infty(\Sigma, \mathbb{C}^*)$ ← complex gauge gp.

$V : \text{Vor}(L,\tau) \to \text{Sym}^d(\Sigma)$, $\quad (\mathcal{L}, \phi) \mapsto \phi^{-1}(0)$?

$$V([A, \phi]) \longmapsto \phi^{-1}(0).$$

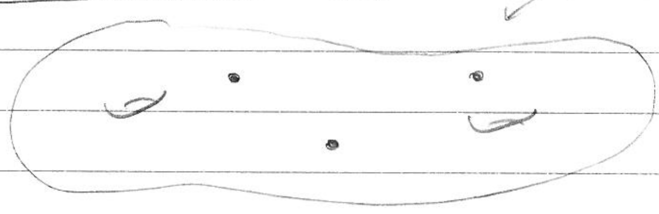
<u>Theorem</u> (Jaffe-Taubes, Bradlow, Garcia-Prada, ...)
The map V is biholomorphic.

2nd equation is a moment map for the Kähler structure this map gives us.

<u>Example</u>
Suppose L is trivial. Take A trivial connection, ϕ constant section with $|\phi|^2 = \tau$.
This is the unique sol^n when $d = 0$.

<u>Solutions of Vortices with $d > 0$</u>

Σ : divisor $\phi^{-1}(0)$.



Interesting structure is all near $\phi^{-1}(0)$. Far from them A is nearly flat, $|\phi|^2 \approx \tau$. One has precise estimates like

$$\left(|F_A| + (|\phi|^2 - \tau)\right)(x) \leq C_1 \exp\left(\frac{-C_2}{\tau} \text{dist}(x, \phi^{-1}(0))\right)$$

Inverse of ν:

Take model solutions on $\mathbb{C}$ with $\phi^{-1}(0) = k \cdot [0]$.
Paste these into Σ near the support of a given divisor.
Use cutoff functions.

This is only an approximate solution. But one has the inverse function theorem, which allows us to use Newton's method to get a true solution.

3 points of view

Vortices $Vor(L, \tau)$ ⟷ {effective degree d divisors}

gauge theory — Kähler geometry

{ d-tuples of points }

combinatorics

Seiberg - Witten Monopoles

Now X is a Riemannian 4-manifold.

Choose $\eta \in \Omega^2_X$, $d\eta = 0$

S a spin^c - structure

Equations $SW(S, g, \eta)$

Fields ϕ spinor $\in \Gamma(S^+)$

A Clifford connection on S^+

Then
$$D_A^+ \phi = 0$$
$$\rho(F_{A^t} + i\eta)^+ = (\phi^* \otimes \phi)_0$$

S is a choice from an $H^2(X; \mathbb{Z})$-torsor.
It consists of (S^+, S^-, ρ), where $S^\pm$ are hermitian $\mathbb{C}^2$-bundles (spinor bundles), & $\rho : T^*X \otimes \mathbb{C} \to Hom_{\mathbb{C}}(S^+, S^-)$ is a linear Clifford multiplication map, satisfying

$$\rho(e)^\dagger \rho(f) + \rho(f)^\dagger \rho(e) = -2g(e,f)\, 1_{S^+}.$$

ρ induces another map $\rho : \Lambda^2 T^* X \otimes \mathbb{C} \longrightarrow End(S^+)$

via $e \wedge f \longmapsto \frac{\rho(e)^\dagger \rho(f) - \rho(f)^\dagger \rho(e)}{2}$

This maps $\Lambda^+ \subseteq \Lambda^2$ to $su(S^+)$ (traceless skew hermitian

hence $i\Lambda^+$ to $i\, su(S^+)$

also $\rho(\Lambda^-) = 0$.

<u>Clifford connection</u> means A makes ρ parallel. It's equivalent to give A^t, the induced $U(1)$-connection in $\Lambda^2 S^+$.

$D_A^+ : \Gamma(S^+) \longrightarrow \Gamma(S^-)$ is the Dirac operator.

Take $(e_1, \ldots, e_4)$ an orthonormal frame for $T^* X$

$$D_A^+ = \sum_{i=1}^{4} \rho(e_i) \circ \nabla_{A, e_i}$$

So equation 1 says ϕ is a harmonic spinor.

$\phi^* \otimes \phi$ refers to an endomorphism of S^+

$(\phi^* \otimes \phi)_0$ is the trace-free part.

Gauge group $\mathcal{G} = C^\infty(X, U(1))$ acts as before, preserving eq^ns.

One has a <u>global</u> $\mathcal{G}$-section, up to constants, by taking A_0 a reference Clifford connection, & imposing

$$d^*_{A_0^t}(A^t - A_0^t) = 0 \quad : \quad \text{Coulomb gauge.}$$

Linearised SW + Coulomb equations look like

$D_A \Psi = 0$ to 0^{th} order

$(d^+ + d^*) a = 0$ " $\quad a \in i\Omega^1$

These are elliptic, with index $\text{ind}(D_A) + \text{ind}(d^* + d^*)$, which one can compute.

Sutured Manifolds 2

Sutured Manifolds

Definition (Gabai '83)

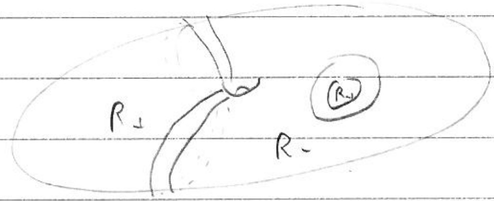
A sutured manifold is an oriented 3-manifold with non-empty boundary divided up as

$$\partial Y = R_+ \cup A \cup R_-$$

(+ive bdary; sutures; -ive boundary)

all subsurfaces of ∂Y meeting only at their boundaries.

e.g



Such that:

- R_+ is oriented like ∂Y
- R_- is oriented opposite to ∂Y
- A is a union of annuli with core curves γ_i oriented like ∂R_+ or ∂R_-
- Each component of Y has non-empty boundary
- Each component of R_+, R_- has non-empty boundary, & meets A.

It is balanced if $\chi(R_+) = \chi(R_-)$

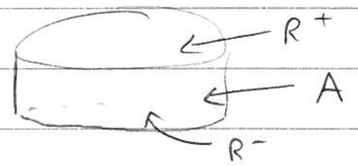
Example:

If Y_0 is closed, take $Y_0 \setminus D^3$. This now have spherical boundary, & we can introduce a trivial suture.

Example

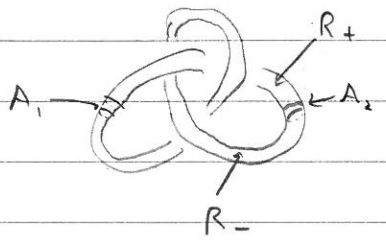
Σ surface with boundary. Consider $\Sigma \times I$.

The boundary naturally composes, e.g. $\Sigma = D^2$



Example

Knot complements: So boundary looks like



(two sutures).

Gabai thought about these in order to study Foliations. One might naturally ask for leaves to be parallel to the boundary on $R_\pm$, transverse to the boundary on A.

It also arises naturally from a contact structure.

Exercise: Show that if Y is a foliated mfld (with orientable foliation) with boundary as above, then it is balanced.

Hint: Think about the first chern class of the 2-plane field.

Definition

A sutured Heegaard diagram is a surface Σ with boundary, & collections $\alpha_1, \dots, \alpha_k$; $\beta_1, \dots, \beta_\ell$ of simple closed curves, with α's & β's pairwise disjoint.

This represents a sutured manifold: take a copy of $\Sigma \times I$ & attach handles for each α_i, β_j. No 0 or 3-handles as we have boundary.

It is balanced iff $k = \ell$.

Grid diagram also gives a sutured manifold by punching a hole for each X & O : it is a diagram for the knot complement with many sutures.

Theorem:

Every sutured manifold has a sutured Heegaard diagram.

Proof

Map $Y \xrightarrow{f} I$ by some generic function so that $f(R_+) = \{1\}$, $f(R_-) = \{0\}$. Pick a gradient-like vector field v parallel to ∂Y on A.

Arrange f to be self-indexing, & to have no index 0 or 3 critical points.

Take $\Sigma = f^{-1}(\frac{1}{2})$, & do standard thing. □

Exercise:

Find a sutured Heegaard diagram for  with as few sutures as possible. (2 is possible).

Definition

Sutured Floer homology $SFH(\mathcal{H})$ is the homology of the chain complex generated by points in $T_\alpha \cap T_\beta$, in $\mathrm{Sym}^k \Sigma$, with differential as before. Discs in $\mathrm{Sym}^k \Sigma$

Example:

$SFH(\Sigma \times I) \cong \mathbb{F}_2$, as there's 1 point in $\mathrm{Sym}^0(\Sigma)$.

Definition:

$\Sigma^2 \subseteq Y^3$ is incompressible if every curve in Σ that bounds a disc in Y also bounds a disc in Σ, & Σ has no sphere components.

Σ is taut if it is incompressible & minimises $\chi_+(\Sigma)$ (minimal genus) in its class in $H_2(Y)$.

For Σ with boundary, $\partial\Sigma \subseteq \partial Y$, have same definition, but minimise $\chi_+(\Sigma)$ within surfaces in the same class in $H_2(Y, \partial\Sigma)$.

A sutured manifold is taut if R_+ & R_- are taut, & Y is irreducible (no non-trivial 2-spheres, or not a connect sum).

Theorem (Gabai '83)

Every taut sutured manifold which is not a rational homology sphere ($b_1 > 0$) is a taut foliation.

Theorem (Juhasz '06)

If Y is a balanced sutured manifold, then

$SFH(Y)$ is non-trivial $\Leftrightarrow$ Y is taut.

In fact

$\dim(SFH(Y)) > 1 \Leftrightarrow Y$ is taut & not a product.

Sutured Decompositions

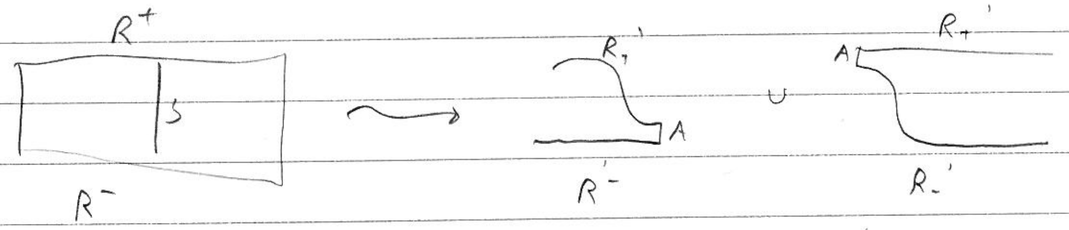
Y sutured manifold, $\Sigma \subseteq Y$ with $\partial\Sigma \subseteq \partial Y$.

Then one can cut along the surface, to get a new mfld with boundary $Y|_S$. New boundary is $S_+ \cup S_-$, distinguished by an orientation.

Get a new sutured manifold with

$$R_+' = (R_+|_{\partial S}) \cup S^+$$
$$R_-' = (R_-|_{\partial S}) \cup S^- \quad \text{smoothing where necessary,}$$

with new pieces of A between.



<u>Theorem</u> (Gabai '83, Scharlemann '89)

If Y is a balanced sutured manifold with $b_1 > 0$, then Y is taut $\iff$ there is a sutured hierarchy

$$Y \searrow_{S_1} Y_1 \searrow_{S_2} Y_2 \searrow \cdots \searrow_{S_n} Y_n \quad \text{sequence of cuts.}$$

where Y_n is a product.

<u>Exercise</u>:

What happens if you decompose along a Seifert surface of a knot?