Heegaard Floer Homology – Lecture 2

Note Title 1/10/2012

due to Ozsváth – Szabó ('00 - '01)

Extension to knots due to Os-Sz & independently by Rasmussen.

Formal Structure – Sketch:

Started w/ attempt at computing Seiberg – Witten

SW : $\begin{pmatrix} \text{closed smooth 4-fold} \\ b_2^+ > 1 \end{pmatrix} \rightsquigarrow$ numbers

SW-invar

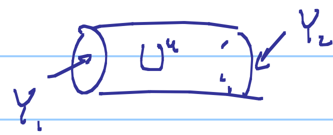
HF attempt to reformulate SW for easier computation. TQFT-ish structure.

Y^3 closed, connected, oriented $\longrightarrow$ • "graded" abelian grp $\widehat{HF}(Y)$

HF = "Floer Homology"

• "graded" $\mathbb{Z}[u]$-modules $HF^+(Y)$, $HF^-(Y)$

$\widehat{HF}(Y)$ is focus of these lectures.
$HF^{\pm}(Y)$ more complicated, needed for 4-folds.

Cobordism (smooth, oriented) 

$\rightsquigarrow \hat{F}_{\omega} : \widehat{HF}(Y_1) \to \widehat{HF}(Y_2)$

Gluing cobordisms $\rightsquigarrow$ composition of morphisms

To get 4-fold invar., can view as cobordism from $\emptyset$ to $\emptyset$, but better, delete two balls and view as cobord from S^3 to S^3. This # is not SW, which vanishes, but rather something 2nd order

For knots, K - knot in Y

$\widehat{HFK}(Y, K)$ - bigraded abelian grp

$\widehat{HFK}_{i,j}(K)$

Note: "Homology" indicates it is the homology of a chain cplx. Not a generalized homology theory.

Technical issues suppressed:

- Spin^c structures
- gradings
- will use $\mathbb{Z}/2$ - coeffs

} all discrete data

Heegaard Diagrams

$$\mathcal{H} = (\Sigma_g, \underbrace{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g}_{\text{circles in } \Sigma})$$

closed, oriented surface

circles in Σ

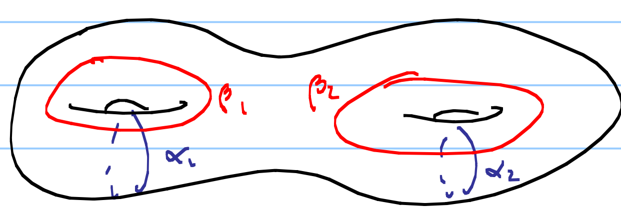
$\alpha_i \cap \alpha_j = \emptyset$

$\beta_i \cap \beta_j = \emptyset$

αs span g-dim'l subspace $\Longleftrightarrow$ αs lin. indep in H^1 $\Longleftrightarrow$ $\Sigma \setminus \bigcup_i \alpha_i$ is connected

βs ______ $\Longleftrightarrow$ βs ______ $\Longleftrightarrow$ ______

e.g.



From Heegaard diagram to $Y^3 =: Y(\mathcal{H})$

- thicken Σ
- attach discs D^2 along αs inside, βs outside
- attach B^3 to fill rest of body

Example above gives S^3.

Fix basepoint $z \in \Sigma \setminus \bigcup_i \alpha \cup \bigcup_j \beta$

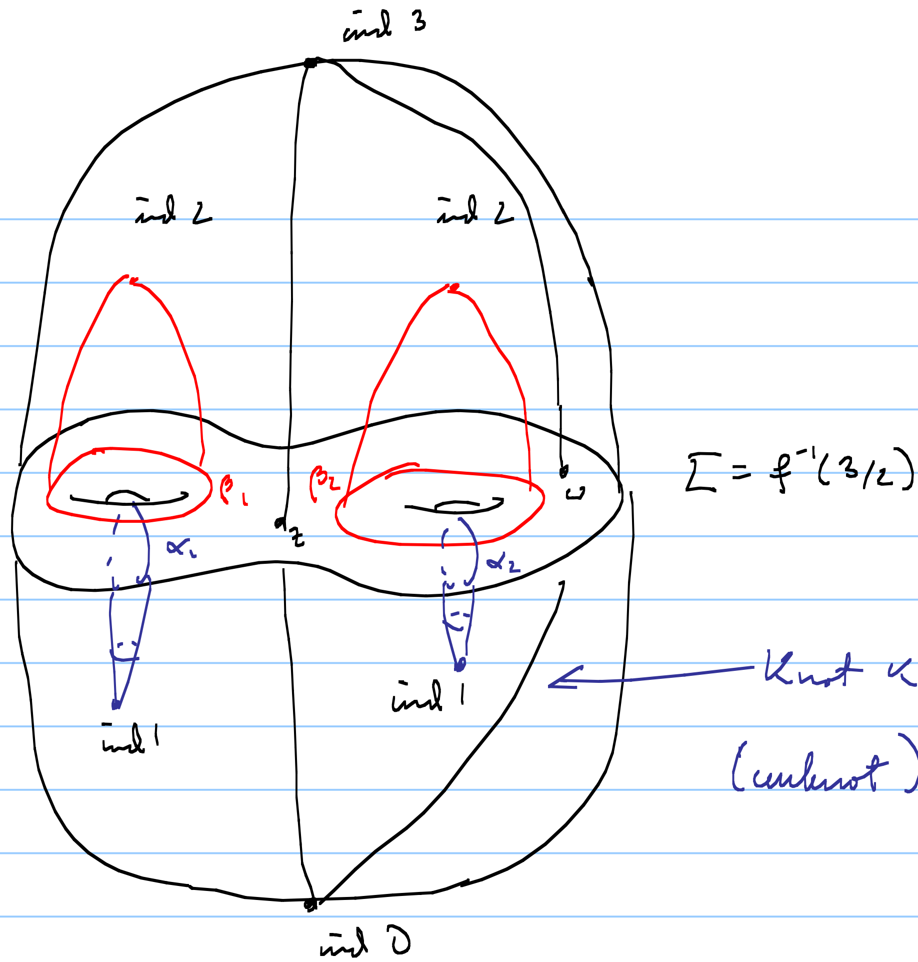
$\rightsquigarrow$ fix a flow line from ind. 0 to ind. 3 crit. point

(Heegaard diagrams come from self-indexing Morse fns on 3-folds, & given Heegaard diagram $\mathcal{H}$, can non-canonically generate a Morse fn on $Y(\mathcal{H})$ giving $\mathcal{H}$).
So have a background Morse fn. here.

This is equiv. to a $B^3 \subset Y = Y(\mathcal{H})$ (via Morse thry)

If we fix a second point $w \in \Sigma \setminus \bigcup_i \alpha_i \bigcup_i \beta_i$ get 2 flow lines from ind. 0 to ind. 3.
Concatenating these gives a knot in Y.

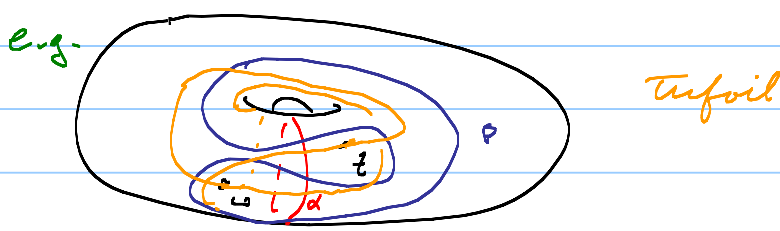
e.g.



$\exists!$ path from z to w in $\Sigma \setminus \alpha$-discs which is unknotted & misses

(not ! in Σ, but is ! once we push it into Y and avoid α-discs).

e.g.



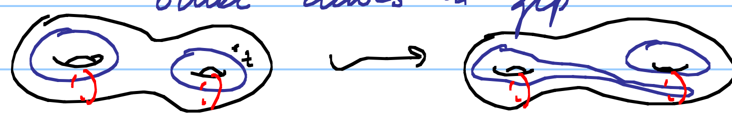
Owns is on presentation of S^3. Want morse fn on S^3 s.t. the knot is a union of flow lines

(T) If $\mathcal{H}$ & $\mathcal{H}'$ represent the same Y^3 (resp. (Y^3, K)), then can get from $\mathcal{H}$ to $\mathcal{H}'$ by a sequence of

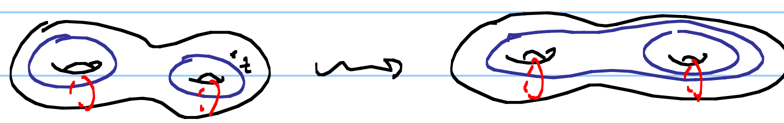
- isotopies
- handle slides
- stabilizations / destabilizations

not crossing z (resp. z & w)

isotopies = slide α & β around w/o intersecting other curves in grp

e.g. 

handleslide:

e.g. 

depends on path connecting one α (or β) to another and thus embedded connect sum along this path.

stabilization = connect sum w/ 

(T) is equivalent to a statement about morse fns. on 3-folds.

Rmk: (T) is an existence proof. Not easy to find such a sequence for two given diagrams.

Fix $\mathcal{H} = (\Sigma, \underline{\alpha}, \underline{\beta}, z)$

$$\underline{\alpha} = \{\alpha_i\}, \quad \underline{\beta} = \{\beta_i\}$$

$$\Sigma^g = \underbrace{\Sigma \times \dots \times \Sigma}_{g}$$

$$\cup$$

$$\alpha_1 \times \dots \times \alpha_g$$

$$\mathrm{Sym}^g(\Sigma) = \Sigma \times \dots \times \Sigma / S_g \quad \sim g\text{-unordered pts in } \Sigma$$

(P) 1) $\mathrm{Sym}^g(\Sigma)$ is a Topological mfld (uses that Σ is 2D)

2) Cx structure J_Σ on Σ induces smooth structure on $\mathrm{Sym}^g(\Sigma)$

3) J_Σ induces cx. structure $\mathrm{Sym}^g(J_\Sigma)$ characterized by $(\Sigma^g, j_\Sigma \times \dots \times j_\Sigma) \longrightarrow (\mathrm{Sym}^g(\Sigma), \mathrm{Sym}^g(J_\Sigma))$ is holom.

Thm (Perutz) $\exists$ Kähler forms compatible w/ $\mathrm{Sym}^g(J_\Sigma)$ s.t. ...

Tori:

$$\begin{array}{ccc} \alpha_1 \times \dots \alpha_g & \subset & \Sigma^g \\ \downarrow \cong & & \downarrow \\ T_\alpha & \subset & \mathrm{Sym}^g(\Sigma) \\ T_\beta & \subset & \end{array}$$

Lagrangian tori

Intersection # (algebraic)

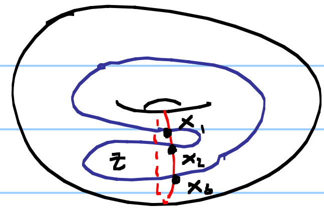
$$T_\alpha \cdot T_\beta = \begin{cases} |H_1(Y)| & \text{if } |H_1(Y)| \text{ finite} \\ 0 & \text{else} \end{cases}$$

Short def: $\widehat{HF}(Y) = HF(T_\alpha, T_\beta \subset \mathrm{Sym}^g(\Sigma, z))$

Lagrangian Floer homology

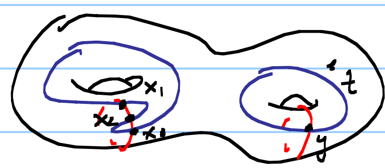
Unpacking this: $\widehat{CF}(Y) = \mathbb{Z}/2\langle T_\alpha \cap T_\beta \rangle$

e.g.: •



$\widehat{CF} = \mathbb{Z}/2\langle x_1, x_2, x_3\rangle$

•



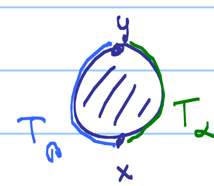
$\widehat{CF}(Y) = \mathbb{Z}/2\langle [x_1,y], [x_2,y], [x_3,y]\rangle$

(D) a Whitney disc from $\underset{\mathbb{T}_\alpha\cap\mathbb{T}_\beta}{x}$ to $\underset{\mathbb{T}_\alpha\cap\mathbb{T}_\beta}{y}$

is a cts. map $D^2 \longrightarrow \mathrm{Sym}^g(\Sigma)$

s.t. $i \longmapsto y$ & $(i,-i) \subseteq \mathbb{T}_\beta$

$-i \longmapsto x$ $\quad (-i,i) \subseteq \mathbb{T}_\alpha$



(D) $\pi_2(x,y) := \{$ htpy classes of Whitney discs from x to $y\}$

(D) A Whitney disc $u: D^2 \longrightarrow \mathrm{Sym}^g(\Sigma)$ is holom. if $du \circ i = \mathrm{Sym}^g(J_\Sigma)\circ du$.

(D) $\varphi \in \pi_2(x,y)$

$\mathcal{M}(\varphi) := \{$ holom Whitney discs in class $\varphi \}$

(P) $\mathcal{M}(\varphi)$ is (generically) a finite dim'l mfld of dimension given by some algebro-topological # called (Maslov) index and written $\mathrm{ind}(\varphi)$ or $\mu(\varphi)$.

Use to define differential on $\widehat{CF}(Y)$:

$$\partial : \widehat{CF}(Y) \longrightarrow \widehat{CF}(Y)$$
$$x \longmapsto \sum_{y \in T_\alpha \cap T_\beta} \ \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mathrm{ind}(\varphi) = 1 \\ \varphi \cap \{z\} \times \mathrm{Sym}^{g-1}(\Sigma) = \emptyset}} \#\left(\mathcal{M}(\varphi)/\mathbb{R}\right) \cdot y$$

importance of this is that w/o it, $\widehat{HF}(Y)$ would only depend on cup product structure on $H^*(Y)$. Would see too little.

(P) $\partial^2 = 0$

$\widehat{HF}(Y) := \ker \partial / \mathrm{im}\, \partial$

(T) $\widehat{HF}(Y)$ is an invariant of Y
(i.e. it's invar. under moves in (T) above)
(up to isom. basepoint doesn't matter)

Sample computations:

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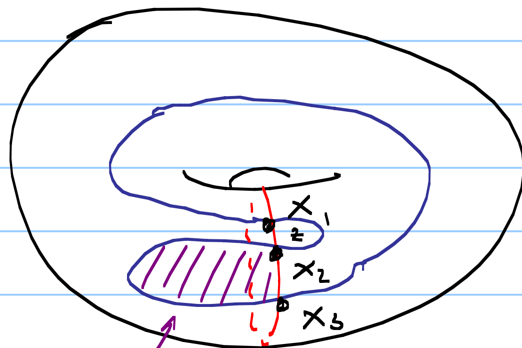


Whitney disc $x_3 \to x_2$

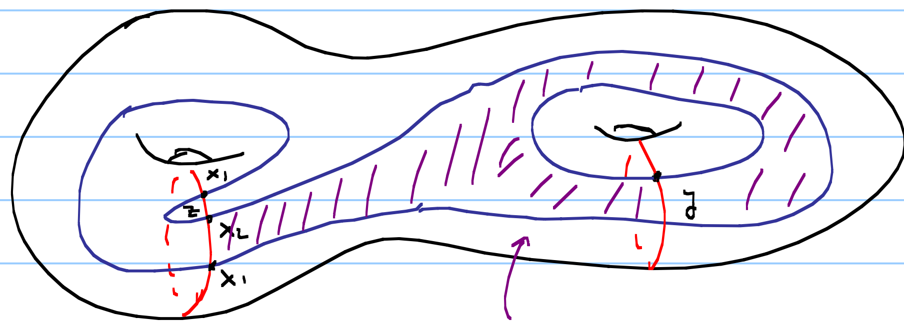
By Riemann mapping thm, $\exists !$ holom u.

Exercise: $\nexists$ other classes of Whitney discs here.

$\Rightarrow \quad \widehat{CF}, \quad \partial : x_3 \longrightarrow x_2$

So $\quad \widehat{HF}(S^3) = \mathbb{Z}/2$

• Hard computation:



corresponds to a holom. Whitney disc in $\mathrm{Sym}^g(\Sigma)$

Riemann surface

$$\begin{array}{ccc} S & \xrightarrow{u_\Sigma} & \Sigma \\ {\scriptstyle u_D}\downarrow & & \\ D^2 & & \end{array}$$

- u_D, u_Σ holom.
- u_D g-fold branched cover

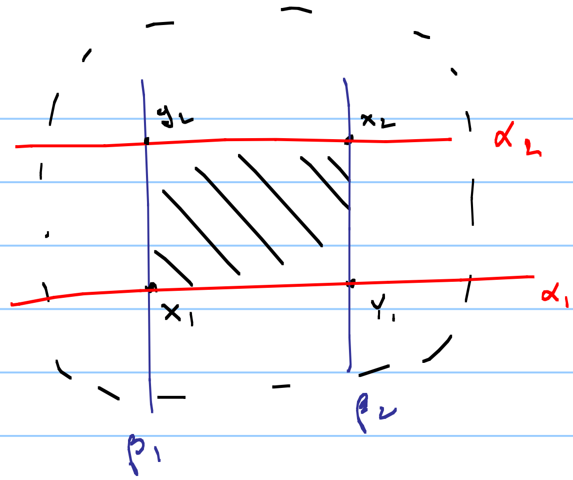
$u_D^{-1}(p) = \{q_1, \dots, q_g\}$, $u_\Sigma(u_D^{-1}(p)) \in \mathrm{Sym}^g(\Sigma)$

So we get map $D^2 \longrightarrow \mathrm{Sym}^g(\Sigma)$

This map will be holom.

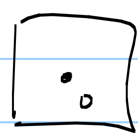
(Can go other way)

e.g.



$$[x_1, x_2] \longrightarrow [y_1, y_2]$$

$S =$



$\longrightarrow \Sigma$

$$\big\downarrow \quad z \mapsto z^2$$

$$\longrightarrow D^2 \longrightarrow \mathrm{Sym}^2(\Sigma)$$