Note Title 1/10/2012

Sutured Mflds

D (Gabai '83)

A sutured mfld (for us) is an oriented 3-mfld Y w/ bdry divided up

$$\partial Y = R_+ \cup A \cup R_-$$

positive, sutures, negative

s.t. • R_+ oriented like ∂Y

R_- ——— $-\partial Y$

• A is a union of annuli

; γ_i, the core curves of the annuli oriented compatibly w/ $\partial R_+ / \partial R_-$

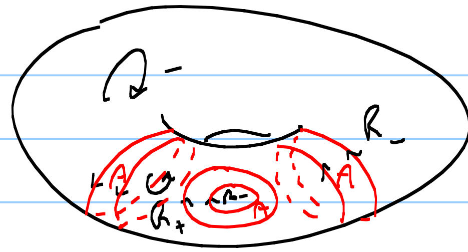
• Each component of Y meets ∂Y

R_+ ——— ∂R_+, A

R_- ———

It's balanced if $\chi(R_+) = \chi(R_-)$

e.g. •



non-balanced

• Y_0 closed

• $Y = Y_0 \setminus D^3$ w/ $R_+ = R_- =$ disc

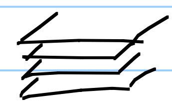
• $\Sigma \times I$, $R_+ = \Sigma \times \{1\}$, $R_- = \Sigma \times \{0\}$, $A = \partial\Sigma \times I$

surface w/ bdry

• Knots: (complement)

Gabai cared about this for:

(Oriented) Foliations



Formally smooth integrable 2-plane field

stack of leaves near each pt.

On Y w/ bdry, natural to ask $\underbrace{\text{parallel to bdry on part of } \partial Y}_{R_\pm}$ & $\underbrace{\text{transverse elsewhere}}_{A}$

$R_\pm$ (signs indicate where bdry agrees w/ foliation orientation)

Sutured mflds also arise naturally from contact structures.

Exercise: If $(Y, \partial Y, \mathcal{F})$ nice as above, show the associated sutured structure on Y is balanced.

oriented foliation

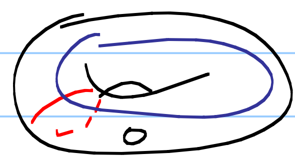
Hint: think about c_1 of 2-plane field

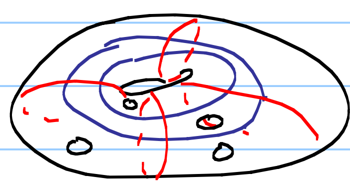
(D) A *sutured Heegaard diagram* is a surface Σ w/ bdry and two collections of simple closed curves $\underline{\alpha} = \{\alpha_i\}_1^k$, $\underline{\beta} = \{\beta_j\}_1^\ell$, w/ $\underline{\alpha}$ ($\underline{\beta}$) disjoint & homologically linearly independent.

$\Sigma \times I$ represents a sutured mfld:

$\Sigma \times I \cup$ 2-handles on $\alpha_i \times \{0\}$ (R_-)
$\beta_i \times \{1\}$ (R_+)

It will be balanced $\Leftrightarrow k = \ell$.

e.g. :  $\rightsquigarrow S^3 \setminus B^3$

grid diagram  $\rightsquigarrow$ knot complement w/ extra sutures

(T) Every sutured mfld has a sutured Heegaard diagram.

Pf: Find $Y \xrightarrow{f} I$ generic s.t. $f(R_+) = \{1\}$, $f(R_-) = \{0\}$. Take gradient-like v. field

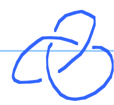
V parallel to ∂ on A.

Choose f s.t. no index $0, 3$ crit pts.

$\mathrm{ind}(p) = 1,\ \mathrm{ind}(q) = 2 \Rightarrow f(p) < f(q)$

$\Sigma = f^{-1}(\frac{1}{2})$.

"$\square$"

Exercise: Find a sutured Heegaard diagram for  w/ as few sutures as possible.

Rmk: Same moves as for Heegaard diagrams apply to sutured Heegaard diagrams.

(D) Sutured Floer Homology $SFH(\mathcal{H})$

$\nwarrow$ sutured Heegaard diagrams

is the homology of the chain cx generated by k-tuples of intersections btw α_i & β_j (i.e. generators are $\in \mathrm{Sym}^k(\Sigma)$)

w/ differential counting discs in $\mathrm{Sym}^k(\Sigma)$, or cylindrical version via surfaces in $\Sigma \times [0,1] \times \mathbb{R}$

Example: $SFH(\Sigma \times I) \cong \mathbb{F}_2$ (in $\mathbb{Z}/2$-coeffs)
(no curves)

(D) $S \subset Y$ is incompressible if every curve in S
surface 3-fold
which bounds a disc in Y also bounds one in S & S has no sphere components.

S is taut if it is incompressible & minimizes
$\chi_+(S) = \begin{cases} \max[0, \chi(S)] & \chi(S) \leq 0 \\ 0 & \text{else} \end{cases}$ w/in homology class of S

For S w/ $\partial S \subset \partial Y$, do same thing,
minimize $\chi_+(S)$ w/in surfaces in same $[S] \in H^2(Y, \partial S)$

A sutured mfld is taut if $R_\pm$ is taut & Y is irreducible as a 3-fold.
(no nontrivial 2-spheres or almost equivalently, not a connect sum).

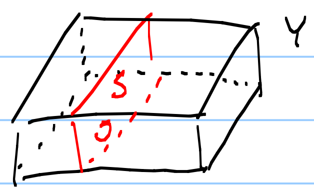
(T) (Gabai '83) Every taut sutured mfld which w/ $b_1^+ > 0$ has a taut foliation
(e.g. if ∂Y has a non-sphere component).

Ⓣ (Juhasz '06)

Y balanced sutured mfld, then $SFH(Y) > 0$
$\iff$ Y is taut.

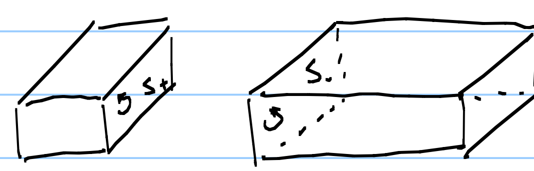
Further, $\dim(SFH(Y)) > 1 \iff Y$ is taut & not a product.

Sutured Decompositions



$S \subset Y$
$\partial S \subset \partial Y$

Cutting $Y|_S$

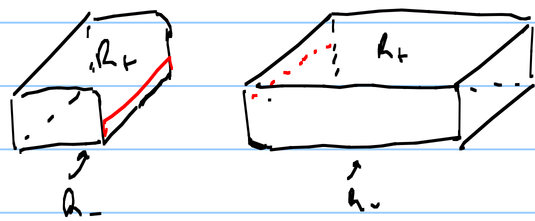


S_+ - part where bdry agrees w/ or. of S
S_- - part where bdry disagrees w/ or. of S

Get a new sutured mfld w/ $R_+' = R_+|_{\partial S} \cup S_+$
$R_-' = (R_-|_{\partial S} \cup S_-)$

(plus some smoothing)

e.g.



T (Gabai '83, Scharlemann '89)

Y - balanced sutured mfld, $b_1 > 0$

Y is taut $\iff$ $\exists$ sutured hierarchy, i.e. a sequence of suture decompositions

$Y_1 \searrow_{S_1} Y_2 \searrow_{S_2} Y_2 \searrow \cdots \searrow Y_n$ where Y_n is a product

Proof fairly constructive.

Exercise: What happens to a knot complement if we take a suture decomposition along a Seifert surface of the knot.