Quantization & Categorification — Gukov 1

Both these concepts have applications in

- Representation Theory $\rightsquigarrow$ Geometric Rep Theory
- (Quantum) low-dimensional Topology
  - knots (A source of examples)

These two are actually connected: one looks at knots coloured by a representation R of a Lie algebra $\mathfrak{g}$. Looking at the unknot involves purely representation theoretic data.

Our starting data will be a knot K, & a rep^n R of $\mathfrak{g}$. To this we want to associate quantum group invariants.

Examples:

$\mathfrak{g} = \mathfrak{sl}(N)$, R the fundamental representation, aka $R = \square$ (a Young tableau, via Schur-Weyl duality).

To this, one associates a quantum $\mathfrak{sl}(N)$ invariant $P_N(q)$. One uses a skein relation

$q^N P_N($  $) - q^{-N} P_N($  $) = (q-q^{-1}) P_N($  $)$

$P_N(q) \in \mathbb{Z}[q, q^{-1}]$, depending on a knot K.

Normalise by

$P_N(\text{unknot}) = P_N($  $) = \dfrac{q^N - q^{-N}}{q - q^{-1}}$

$$= q^{-(N-1)} + q^{-(N-3)} + \dots + q^{(N-1)}$$

$\longmapsto N$ under $q \mapsto 1$

Say $\dfrac{q^N - q^{-N}}{q - q^{-1}} = \dim_q R$.

Example: Hopf link  $= K$

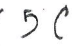
Skein relation allows us to say

$q^N P_N($  $) - q^{-N} P_N($  $) = (q - q^{-1}) P_N($  $)$

$q^N P_N($  $) = q^{-N}\left(\frac{q^N - q^{-N}}{q - q^{-1}}\right)^2 + q^N - q^{-N}$

In general, $P(K_1 \sqcup K_2) = P(K_1) \cdot P(K_2)$.

Remark: Notice that the N only occurs as q^N in our definition. Thus instead, one can trade q, N for $q, a = q^N$. Get a skein relation like

$a P($  $) - a^{-1} P($  $) = (q - q^{-1}) P($  $)$

&

$$P(\bigcirc) = \frac{a - a^{-1}}{q - q^{-1}}$$

These are actually the relations for another invariant: the HOMFLY polynomial. Though it's not a polynomial anymore: get polynomials via

$a = q^N$: quantum $sl(N)$ invariant

$a = q^2$: Jones polynomial

$a = 1$: Alexander polynomial.

Caution: Setting $a = 1$ kills $P(\bigcirc)$. One must choose another normalisation. There are several possibilities

$$P(\bigcirc) = \frac{a - a^{-1}}{q - q^{-1}} \quad \text{as above}$$

or $= 1$

"un-normalised" & "normalised" definitions.

Exercise: Compute normalised & un-normalised HOMFLY polynomial for the trefoil.
Specialise to $a = q^2$, $a = q$, $a = 1$.

Remark

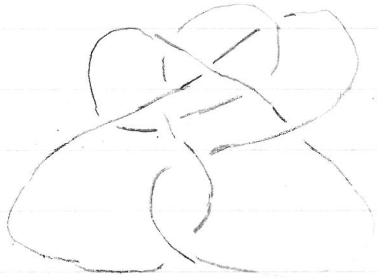
Sometimes, people also use another convention:

$a \longmapsto a^{1/2}$, $q \longmapsto q^{1/2}$, or

$a \longmapsto a^{-1}$, $q \longmapsto q^{-1}$, or both.

Motivating Example:

5_1



10_{132}

In Rolfsen classification

These knots have the same invariants, for all those we've described so far: same HOMFLY polynomial. They are – however – distinguishable...

Additional Reading

- Atiyah – "The geometry & physics of knots" (1990)
- T. Kohno – "Conformal field theory & topology" (2002)
- Cooper, Culler, Gillet, Long, Shalen – Inventiones 118 ('94) 47-84

Classical A-polynomial

Let K be a knot, & let $M = S^3 \setminus \mathrm{Tub}(K)$, a 3-manifold with toral boundary. An obvious knot invariant is $\pi_1(M)$.

Example:
If K is the trefoil, $\pi_1(M) = \langle a, b \mid aba = bab \rangle$

We might study representations $\pi_1(M) \hookrightarrow SL(2, \mathbb{C})$. We'll use this to define an invariant, which will be a plane algebraic curve C given by $A(x, y) = 0$.

Produce elements in $\pi_1(M)$ from the generators l, m of $\pi_1(\partial M) = \pi_1(T^2) = \mathbb{Z}^2$. One can conjugate the images to upper triangular matrices

$$\rho(m) = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(l) = \begin{pmatrix} y & * \\ 0 & y^{-1} \end{pmatrix}$$

Simultaneously, as m & l commute.

E.g.: In our presentation for the trefoil, one has $m = a$, $l = ba^2ba^{-4}$

Example
K the unknot, so M is a solid torus. In the knot complement, one of l & m becomes contractible.

So not every $(x, y) \in \mathbb{C}^* \times \mathbb{C}^*$ comes from a rep of $\pi_1(M)$. Indeed, we're forced to have $y = 1$.

So $A(x,y) = y-1$ describing a curve inside $\mathbb{C}^\times \times \mathbb{C}^\times$: an affine (punctured) line

__Example__ (trefoil)

The abelianisation of $\pi_1(M)$ is just $\mathbb{Z}$.
So an abelian rep forces $y=1$ as above.

This is part of a general phenomenon: $H_1(M) = \mathbb{Z}$ for __any__ knot complement M.

$\therefore$ $A(x,y)$ is always divisible by $(y-1)$: the factor comes from abelian reps.

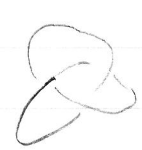
For the trefoil, one gets $A(x,y) = (y-1)(y+x^6)$.

Generally, $\tilde{C} = \{\rho : \pi_1(M) \to SL(2,\mathbb{C})\}/_{\text{conj}}$ is the character variety of $\pi_1(M)$, & we look at the image in the character variety of $\pi_1(\partial M)$.

__Properties__:

- If K is a hyperbolic knot, then $A(x,y) \neq y-1$
- If K is any knot, $A(x,y)$ contains only __even__ powers of x.
- $A(x,y)$ is __reciprocal__, i.e. $A(x,y) = x^a y^b A(x^{-1}, y^{-1})$, i.e. C sits inside $\mathbb{C}^\times \times \mathbb{C}^\times / \mathbb{Z}_2$ (not surprising).
- A can distinguish __mirror__ knots, i.e. reflection in underlying space.

left-handed,



right-handed

$$A(x,y)=0 \longleftrightarrow A(x^{-1},y)=0$$

- A has integer coefficients
- A is **tempered**, i.e. the faces of the Newton polygon define cyclotomic polynomials in 1 variable

e.g

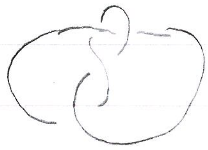
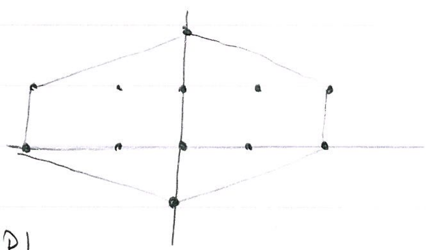


figure 8 knot

Newton polygon



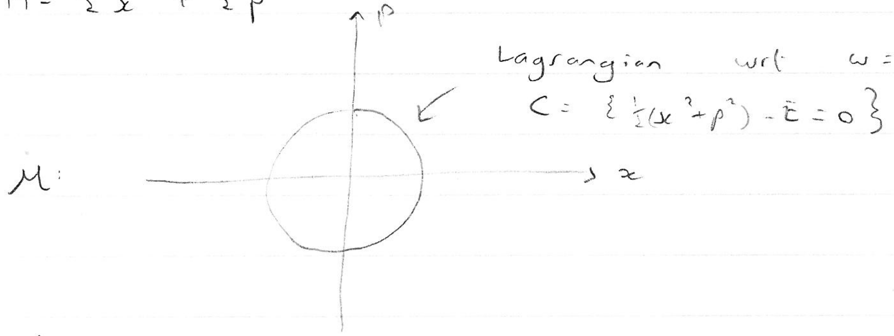
Connection to Physics

- The curve C should be viewed as a holomorphic Lagrangian submanifold in $\mathbb{C}^\times \times \mathbb{C}^\times$
- Its quantisation with symplectic form $\frac{dx}{x} \wedge \frac{dy}{y}$ leads to interesting wave functions.
- It has all the features to be an analogue of the Seiberg-Witten curve in 3d.

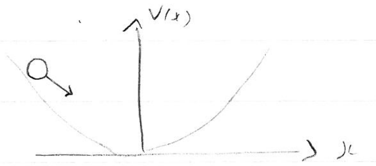
Today: arXiv: 0306165
1003.4808

Quantisation:

$H = \frac{1}{2}x^2 + \frac{1}{2}p^2$



This is a basic example of the harmonic oscillator: H describes a particle oscillating in a quadratic potential



Describe the situation by a particle moving on a circle in the (symplectic) phase space $\mathcal{M}$ above. The circle is a Lagrangian: constant energy slice.

Definition:

A submanifold C in symplectic $\mathcal{M}$ is Lagrangian if $\omega|_C = 0$.

This means we can introduce Liouville form $\theta = d^{-1}\omega$ (the primitive) locally on C. In our example

$$\theta = p\,dx.$$

In the quantum theory, the quantum harmonic oscillator has energy levels

$$\begin{aligned} E &= \hbar(n + \tfrac{1}{2}) \\ &= \frac{1}{2\pi}\int_{\text{Disc bounded by } C} dp \wedge dx \\ &= \oint_C \theta. \end{aligned}$$

We can look at wavefunctions in various states labelled by n. They'll look like

ϑ have form

$$Z(x) \underset{\hbar \to 0}{\approx} \exp\left(\frac{i}{\hbar}\int_0^x \vartheta + \cdots\right)$$

$$= \exp\left(\frac{i}{\hbar}\int \sqrt{2E - x^2}\, dx + \cdots\right)$$

Let's evaluate the ground state. Put $E = 0$

$$\approx \exp\left(-\frac{1}{2\hbar} x^2 + \cdots\right) \qquad \underline{\text{Gaussian}}.$$

This is a <u>semi-classical approximation</u>. In the quantum world, have operators

$$x, p \longmapsto \hat{x}, \hat{p} \quad \text{such that} \quad [\hat{p}, \hat{x}] = -i\hbar$$

<u>e.g.</u>: realise
$$\hat{x}\, f(x) = x\, f(x)$$
$$\hat{p}\, f(x) = -i\hbar \frac{df}{dx}(x)$$

We had classical constraint $C: \frac{1}{2}(x^2 + p^2) - E = 0$.
In the quantum world, this is promoted to an operator equation:

$$\left(\tfrac{1}{2}(\hat{x}^2 + \hat{p}^2) - E\right) Z(x) = 0.$$

This equation has square integrable solutions only for special values of E. Precisely

$$E = \hbar\left(n + \tfrac{1}{2}\right) \qquad n \in \mathbb{Z}_{\geq 0}$$

For $n = 0$, have exact solution

$$Z(x) = \exp\left(-\frac{1}{2\hbar} x^2\right)$$

as in the semi-classical approximation.

More generally

Quantisation should start with a symplectic manifold $(\mathcal{M}, \omega)$, & associate to it a Hilbert space $\mathcal{H}$: we'll draw a table:

$$(\mathcal{M}, \omega) \longrightarrow \mathcal{H} \quad \text{Hilbert space}$$

$$\text{algebra of functions} \longrightarrow \text{algebra of operators}$$

$$f_i \mapsto \hat{f}_i$$

$$\text{Lagrangian submanifolds} \longrightarrow \text{vectors / wavefunctions}$$

$$\mathcal{C} \subseteq \mathcal{M} \mapsto Z \in \mathcal{H}$$

$$\text{Lagrangian } \{f_i = 0\} \mapsto \hat{f}_i Z = 0$$

Not every $(\mathcal{M}, \omega)$ is nicely quantisable, & not every Lagrangian is a satisfactory state.

In general, in the semiclassical limit, we can write

$$Z(x) = \exp\left(\frac{i}{\hbar}\int_0^x \theta + \cdots\right).$$

Remark:

In reality, $\hbar$ is a parameter, but not a formal parameter: we should be able to plug in a real value > 0. In deformation quantisation we take it to be a formal parameter, but this is only an approximation to the quantisation we'd like.

Chern – Simons Theory

M a 3-manifold, e.g. knot complement

- Functional

$$\frac{1}{\hbar}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

where A is a connection on a G-bundle $E \to M$.

We want to extremise this: find flatness equation $dA + A \wedge A = 0$

This is a TQFT, so we should be able to describe it as a functor

Geometry	classical Chern-Simons	Quantum Chern-Simons
2-manifold Σ	symplectic manifold $\mathcal{M} = \mathcal{M}_{Flat}(G, \Sigma)$	vector space $\mathcal{H}_\Sigma$
3-manifold M, $\partial M = \Sigma$	Lagrangian submanifold $\mathcal{C}$ of flat connections on Σ extending to M.	vector $Z(M) \in \mathcal{H}_\Sigma$

Remarks

- One can also think $\mathcal{M} = \{ \pi_1(\Sigma) \to G \} / \text{conj}$ with symplectic form
$$\omega = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}(\delta A \wedge \delta A)$$
Atiyah-Bott noticed this way before Chern-Simons theory was studied in this way.

- One can also think $\mathcal{C} = \{ \pi_1(M) \to G \} / \text{conj}$
$= \text{zero locus of A-polynomials}$

get multiple polys if Σ has genus >1.

So quantisation should produce $\hat{A}_i$ with $\hat{A}_i Z = 0$

Exercises:

- Check $\omega|_\mathcal{C} = 0$
- If $g > 1$, show $\dim \mathcal{M} = (2g-2) \dim G$

Now, let M be a knot complement as before, $\partial M = \Sigma = T^2$. Fix $G = SL(2, \mathbb{C})$. Our $\mathcal{C}$ is exactly the $C \subseteq \mathbb{C}^* \times \mathbb{C}^* / \mathbb{Z}_2 = \mathcal{M}$ we had before: zero locus of the A-polynomial. Let's quantise.

First, in this example, $\omega = \frac{dy}{y} \wedge \frac{dx}{x}$

Promote x, y to operators $\hat{x}, \hat{y}$ s.t.

$$\hat{y}\,\hat{x} = q\,\hat{x}\,\hat{y}$$

where $q = e^{\hbar}$.

Like before, let $\hat{x} f(x) = x f(x)$

$= q^{n} f(n)$ by passing to $n = \frac{1}{\hbar}\log(x)$.

$$\hat{y} f(n) = f(n+1) \quad \text{(shift)}$$

satisfying the commutation relation.

$$A(x, y) = 0 \longrightarrow \hat{A}(\hat{x}, \hat{y}; q)$$

Write $A(x, y) = \sum a_k(x)\, y^k$

Then $\hat{A}(\hat{x}, \hat{y}; q) = \sum_k a_k(x; q)\, \hat{y}^k$

Example:

$K =$ Trefoil. Then $A(x, y) = (y^{-1} - 1)(y + x^3)$

$$\longrightarrow \hat{A}(\hat{x}, \hat{y}; q) = \alpha\, \hat{y}^{-1} + \beta + \gamma\, \hat{y}$$

where

$$\alpha = \frac{x^2(x - q)}{x^2 - q}$$

$$\beta = q\left(1 + x^{-1} - x + \frac{q - x}{x^2 - q} - \frac{x - 1}{x^2 q - 1}\right)$$

$$\gamma = \frac{q - x^{-1}}{1 - q x^2}$$

In classical limit $q \longrightarrow 1$, this devolves to

$$\alpha = \frac{x^2}{1 + x}$$

$$\beta = \frac{1 - x^3}{x(1 + x)}$$

$$\gamma = \frac{-1}{x(1 + x)}$$

The equation $\hat{A} Z = 0$ is nothing but

$$0 = \alpha(q^n; q) Z_{n-1} + \beta(q^n; q) Z_n + \gamma(q^n; q) Z_{n+1}$$

Exercise:

Try to solve this with initial conditions

$$Z_n = 0 \quad \text{if} \quad n \leq 0$$

$$Z_1 = 1$$

To determine $Z_n(q)$ for $n \geq 2$.

Solution:

Given $\alpha(q^n, q)\, T_{n-1} + \beta(q^n, q)\, T_n + \gamma(q^n, q)\, T_{n+1} = 0$

$T_n = 0 \quad n \leq 0$

$T_1 = 1$

We find

$T_2 = q + q^3 - q^4$ normalised Jones polynomial

$T_3 = q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11} \ldots$

How to turn polynomials into q-difference operators:

$A(x, y) \rightsquigarrow \hat{A}(\hat{x}, \hat{y}\,; q) = A(\hat{x}, \hat{y}) + \hbar A_1(\hat{x}, \hat{y}) + \ldots$

References: arXiv: 1205.2261

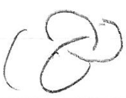
1108.0002

1102.4847

describing general machinery

Our $T_n \in \mathbb{Z}[q, q^{-1}]$ are the so-called "n-coloured Jones polynomials": generalisations of the Jones polynomial, where the knot is decorated by a representation V_n of $sl(2)$ of dimension n.

We can also solve Wednesday's exercise, & compute the HOMFLY polynomial of the trefoil to be

P () $= \left(\frac{a - a^{-1}}{q - q^{-1}}\right)(a^2 q^{-2} + a^2 q^2 - a^4)$

un-normalised. we often normalise & divide powers by two.

Categorification:

One thing this might mean is upgrading TQFTs, such that one recovers the original theory by dimensional reduction, e.g.

$3d \text{ TQFT} \longrightarrow 4d \text{ TQFT}$

We can compare:

Geometry	3d TQFT	$\xrightarrow{\text{Categorification}}$ 4d TQFT
3-manifold M	number $Z(M)$	vector space $\mathcal{H}_M$
knot $K \subseteq M$	$P(K)$	
2-manifold Σ	vector space $\mathcal{H}_\Sigma$	category $\mathcal{C}_\Sigma$

In this vein, we can categorify our invariants

$$\begin{tikzcd}
 & \text{HOMFLY homology} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\text{HOMFLY polynomial}\\ P(a,q)\end{array} \arrow[dr, "a=q^2"] & & \begin{array}{c}\text{Khovanov homology}\\ Kh(K)\end{array} \arrow[dl] \\
 & \begin{array}{c}\text{Jones polynomial}\\ J(q)\end{array} &
\end{tikzcd}$$

We'll spend the rest of the lecture explaining this.

Khovanov homology

Associated to the Lie algebra $sl(2)$, & its representation V_2. It is a bigraded complex $\mathcal{H}_{ij}$, and

$$\sum_{i,j} (-1)^i q^j \dim \mathcal{H}_{ij}(K) = J(q)$$

recovers the Jones polynomial. We can introduce the Poincaré polynomial of the complex

$$P^{sl(2),V_2}(q,t) = \sum_{i,j} t^i q^j \dim \mathcal{H}_{ij}(K)$$

recovering Jones by setting $t=-1$.

Example:

For the trefoil, $P^{sl(2),V_2}(K) = q + q^3 t^2 - q^4 t^3$

HOMFLY homology

Now have $\mathcal{H}_{ijk}(K)$ tri-graded, and

$$\sum_{i,j,k} (-1)^i q^j a^k \dim \mathcal{H}_{ijk}(K) = P(a,q)$$

recovers the HOMFLY polynomial

Again, we can introduce the Poincaré polynomial

$$\mathcal{P}(a,q,t) = \sum_{i,j,k} t^i q^j a^k \dim \mathcal{H}_{ijk}(K).$$

Our polynomials are now related by

$$\begin{tikzcd}
 & \mathcal{P}(a,q,t) \arrow[dl, "t=-1"'] \arrow[dr, "a=q^2"] & \\
P(a,q) \arrow[dr, "a=q^2"'] & & P^{sl(2),V_2}(q,t) \arrow[dl, "t=-1"] \\
 & J(q) &
\end{tikzcd}$$

← a little more subtle.

There is a coloured version of each theory, & the colour version is also captured by an algebraic curve: a & t - deformation of A polynomial.

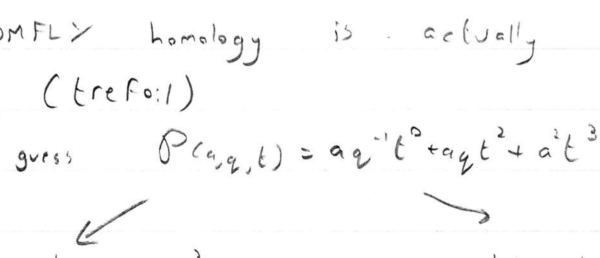
This HOMFLY homology is actually computable.

<u>Example</u> (trefoil)

We can guess $\mathcal{P}(a,q,t) = aq^{-1}t^0 + aqt^2 + a^2t^3$

$$\begin{tikzcd}
 & \mathcal{P}(a,q,t) \arrow[dl] \arrow[dr] & \\
P(a,q) = aq^{-1} + aq - a^2 \arrow[dr] & & Kh(q,t) = q + q^3t^2 + q^4t^3 \arrow[dl] \\
 & J(q) = q + q^3 - q^4 &
\end{tikzcd}$$

We could've computed this even <u>without</u> knowing the Khovanov homology. First, write the answer pictorially as



We assign <u>t-degree</u> to each term in the HOMFLY. Do this by computing the HOMFLY polynomial at $q = a$. We get a single monomial, as it is an $sl(1)$ - theory. The $sl(1)$ homology must be trivial too

<u>Also</u>, setting $a = q^{-1}$ yields a single monomial in HOMFLY.

<u>Really</u>, the map HOMFLY homology $\longrightarrow$ Khovanov, or more generally $sl(N)$ homology, is taking homology with respect to one differential, then collapsing a grading via $a = q^N$.

$\mathcal{H}_{\mathrm{fin}}$ comes with differentials

d_N of degree $(-1, N, -1)$ if $N \geqslant 0$

d_N of degree $\underset{a \quad\; q \quad\;\; t}{(-1, N, -3)}$ if $N < 0$

These can be computed, allowing computations of HOMFLY homology. Often even works for coloured HOMFLY homology.