Notre Dame Summer School — Henriques 1

Extended Conformal Field Theory

Idea goes back to Freed in the 90's: TFT after Atiyah. Extended CFTs first proposed by Stolz & Teichner

We'll start with Segal's definition of a CFT:
disclaimer: this is not the only def^n.

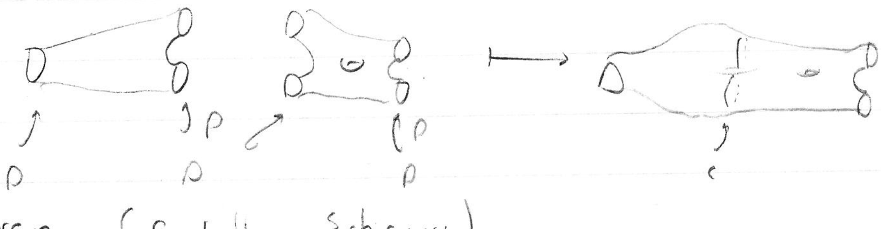
Definition (Segal)

A CFT is a symmetric monoidal functor between the following categories:

i) ob: 1d compact oriented smooth manifolds
mor: cobordisms equipped with a complex structure in the interior.
$\otimes = \amalg$

ii) ob: Hilbert spaces
mor: bounded linear maps
$\otimes = \hat{\otimes}$

No identities in i), but this doesn't matter. Can get away without, or formally adjoin them. Composition requires more care. One can certainly glue in the obvious way as a space



Theorem (Randal-Williams?, Schiggers)

There is a unique (smooth &) complex structure on the glued manifold, extending those on the pieces

This operation is called "<u>conformal welding</u>".
Proving this one uses .

<u>Lemma</u>:

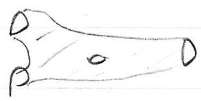
If $D \subseteq \mathbb{C}$ with C^∞ boundary is simply connected, & f is a Riemann map to the standard unit disc, then f is C^∞ all the way to ∂ (not just holo on interior).

If one has a diffeo $S^1 \longrightarrow S^1$, & one uses it to weld unit disc to itself, one produces $\mathbb{P}^1$ with an embedded C^∞ curve.

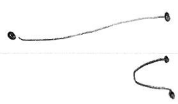
<u>Extended version</u>: replace categories by 2-categories

<u>Picture</u>:

 $\longmapsto$ number

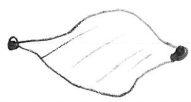
 $\longmapsto$ linear map

 $\longmapsto$ Hilbert space

 $\longmapsto$ bimodule

• $\longmapsto$ Von Neumann algebra

& in between

 $\longmapsto$ linear map of bimodules

One could, instead of talking about algebras, talk about their categories of representations. We can make this precise in the following way.

2-category of conformal surfaces

everything is oriented

	morphisms	structure	local model
0	0-manifolds M	✓	pt
1	1-cobordisms $M_0 \hookrightarrow W \hookleftarrow M_1$ with collars $M_0 \times [0,\varepsilon) \to W$, $M_1 \times (1-\varepsilon, 1] \to W$	C^∞ away from boundary	$[0,1]$
2	2-cobordisms $W_0 \hookrightarrow \Sigma \hookleftarrow W_1$ with compatibility between collars at the ends	conformal away from boundary	

for $f, g \in C^\infty([0,1], \mathbb{R})$, $f \geq g$, $f = g$ near 0 and 1

$\Sigma = \{(x,y) : f(x) \leq y \leq g(x)\}$

Examples:

- 0d: only $\bullet_+$, $\bullet_-$ & unions there of. $\pm$ denote orientations.

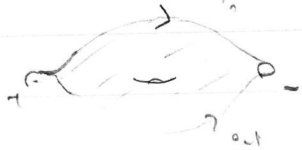
- 1d: write $\circ$ for input points $\bullet$ for output

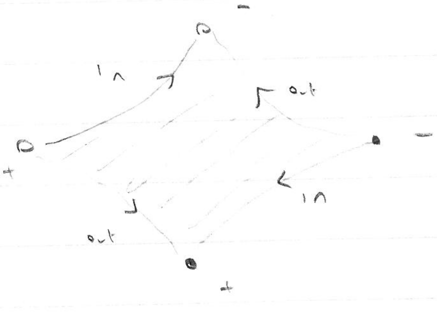
morphism $\bullet_+ \to \bullet_+$

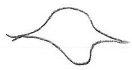
morphism $\bullet_+ \amalg \bullet_- \to \varnothing$

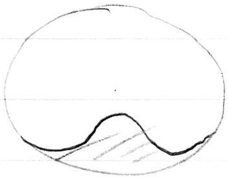
- 2d:

,

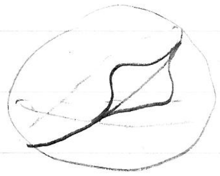




Work needs to be done to glue: firstly, must glue local models  along edges.
One does this by embedding our local models in discs



& gluing along the whole S^1 boundary, yielding our glued local models inside a $\mathbb{P}^1$:



Then use what we already discussed. Still get conformal structures as before.

Now, the target category.

Definition

A von Neumann algebra is a topological $*$-algebra $/\mathbb{C}$ that can be embedded in $B(\mathcal{H})$, for $\mathcal{H}$ a Hilbert spaces as a closed sub-$*$-algebra.

embedding is not part of the data

The relevent topology on $B(\mathcal{H})$ wrt which this is closed is called the ultra-weak topology. There is a pairing

$$B(\mathcal{H}) \times \text{Trace-class ops} \longrightarrow \mathbb{C}$$
$$(a, b) \longmapsto \mathrm{Tr}(a, b)$$

inducing this topology on $B(\mathcal{H})$.

A module is a choice of $\mathcal{H}$ & cts hom $A \to B(\mathcal{H})$. An (A, B)-bimodule is a pair of homs $A \to B(\mathcal{H})$, $B^{op} \to B(\mathcal{H})$ with commuting images

$*$-algebra hom

One might attempt to define the relevent Morita 2-category. Composite is

$$ {}_A M_B \circ {}_B N_C \overset{def}{=} {}_A M \otimes_B N_C . $$

This works literally for usual algebras, but more work must be done for von Neumann algebras.

We first see things are complicated because a vN algebra A is <u>not</u> a bimodule over itself. One must feed A into a machine, outputting a Hilbert space $L^2 A$ which is an (A,A)-bimodule. <u>This</u> is the unit.

<u>Composition</u> (Connes Fusion)

If M is a right A-module, N a left A-module, put $M \boxtimes_A N = M \otimes_A (\mathrm{Hom}_A(L^2 A, N))$ (completed suitably). One can check $L^2 A$ is an identity for this

Comes from saying

$$ M \cong M \boxtimes_A L^2 A \xrightarrow{1 \times f} M \boxtimes_A N $$

So $(m, f) \in M \boxtimes_A N$ if $m \in N$, $f \; L^2 A \to N$ & turning it around.

More symmetrically

$$ M \boxtimes_A N \cong \mathrm{Hom}(L^2 A, M) \otimes_A L^2 A \otimes_A \mathrm{Hom}(L^2 A, N) $$

Henriques 2

There are two distinct things called CFTs. Full CFTs as discussed yesterday, and Chiral CFTs, which are different.

e.g: Vertex operator algebras & conformal nets are formalisms describing chiral CFTs.

Chiral CFTs are an intermediate step towards full CFTs. Loop groups provide examples of CFTs.

The additional data to go from a Chiral CFT to a full CFT is the structure of a Frobenius algebra object. One can actually do better, & construct an extended full CFT from this data.

Chiral CFTs:

We'll give a Segal style formalism:

assign

1 manifold $\longmapsto$ category $\mathcal{C}$

& functor $\mathcal{C} \longrightarrow$ Hilb

bordism Riemann surface $\longmapsto$ functor $F: \mathcal{C}_{in} \to \mathcal{C}_{out}$

$\lambda \in \mathcal{C}_{IN} \longmapsto H_\lambda \to H_{F(\lambda)}$ linear map of Hilbert spaces.

satisfying some axioms

Example

G a Lie group, LG loop group

1-manifold $\longmapsto$ Rep(LG)

Riemann surface $\longmapsto$ push-pull via moduli of G-bundles

This has not been fully constructed.

To construct full CFTs, we'll instead use the <u>conformal net</u> formalism. We'll define them properly tomorrow. But today we'll describe the data.

Conformal Nets

Have the data of a functor

$$\mathcal{A} : \left\{ \begin{array}{c} \text{compact oriented} \\ \text{1-manifolds with } \partial \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{von Neumann} \\ \text{algebras} \end{array} \right\}$$

morphisms on the LHS are <u>embeddings</u>.

If $I \overset{j}{\hookrightarrow} J$, either j is orientation preserving or reversing (require this)

$$\text{Preserving} \longmapsto \mathcal{A}(I) \hookrightarrow \mathcal{A}(J) \quad \text{injective}$$
$$\text{reversing} \longmapsto \mathcal{A}(I) \hookrightarrow \mathcal{A}(J)^{op}$$

On RHS morphisms are homomorphisms or antihomomorphisms. Both cats, and $\mathcal{A}$, are partially monoidal.

Example

- G a connected compact Lie group
- k a <u>level</u>. If G is simple, $k \in \mathbb{N}$

  more generally, a choice of bi-invariant metric s.t. square length of geodesics are integers.

If I is a 1-manifold, define the group

$$L_I G = \mathrm{Map}_*(I, G) \quad \text{: boundary points of } I \text{ map to } e \in G.$$

& all derivatives at those points vanish.

$L_I G$ is a nice group, with a central extension given by a cocycle c. If $f, g \in \mathrm{Lie}(L_I G)$, then

$$c(f, g) = \int_I \langle f, dg \rangle_k \qquad \langle , \rangle_k \text{ metric.}$$

Now $\mathcal{A}_{LG,k}(I) :=$ Completion of the group algebra of $L_I G$ with multiplication twisted by c.

nearly the same as the completed group algebra of the central extension, but you have to identify the central S^1 with the S^1 in the scalars.

More precisely, given a cocycle c, we modify multiplication on $\mathbb{C}G$ to

$$g \cdot_c h := c(g,h)\, g \cdot h$$

Representations of Conformal Nets

Definition

A representation of a conformal net $\mathcal{A}$ is a Hilbert space H equipped with compatible actions of $\mathcal{A}(I)$ for every $I \subsetneq S^1$.

In good cases, this is the same as an action of $\mathcal{A}(S^1)$.

Example: $G = SU(2)$

Recall $SU(2)$ has irreps $V_0, V_1, V_2, \ldots$, $\dim V_k = k+1$,

$$V_i \otimes V_j = V_{|j-i|} \oplus V_{|j-i|+2} \oplus \cdots \oplus V_{j+i}$$

$LSU(2)_k$ has irreps $V_0, V_1, V_2, \ldots, V_k$ ∞-dim^l Hilbert spaces, but $\otimes$ is still determined by

$$V_1 \otimes V_n = \begin{cases} V_1 & n=0 \\ V_{n-1} \oplus V_{n+1} & n \neq 0,k \\ V_{k-1} & n=k \end{cases}.$$

Exercise: From this rule, compute $V_i \otimes V_j$

Say now $\mathcal{A}$ is the corresp. conformal net, & H, K are two representations.

Works for any $\mathcal{A}$

Look at half circles:

& take the reflection diffeomorphism ϕ between these. $\mathcal{A}(I) \curvearrowright H$, $\mathcal{A}(J) \curvearrowright K$, & ϕ induces an isomorphism $\mathcal{A}(J) \xrightarrow{\sim} \mathcal{A}(J)^{op}$, as orientation reversing. Thus think of H as an $(\mathcal{A}(J), \mathcal{A}(I))$-bimodule, & form

$$H \boxtimes_{\mathcal{A}(I)} K,$$

a bimodule for the remaining half circles.

<u>Frobenius algebra objects</u>: In the category of representations (unitary version). Object $Q \in \mathcal{C} = Rep(\mathcal{A})$ equipped with a multiplication, unit, comultiplication, counit, satisfying usual Frobenius algebra conditions.

We'll also impose <u>extra conditions</u>: Pictorially

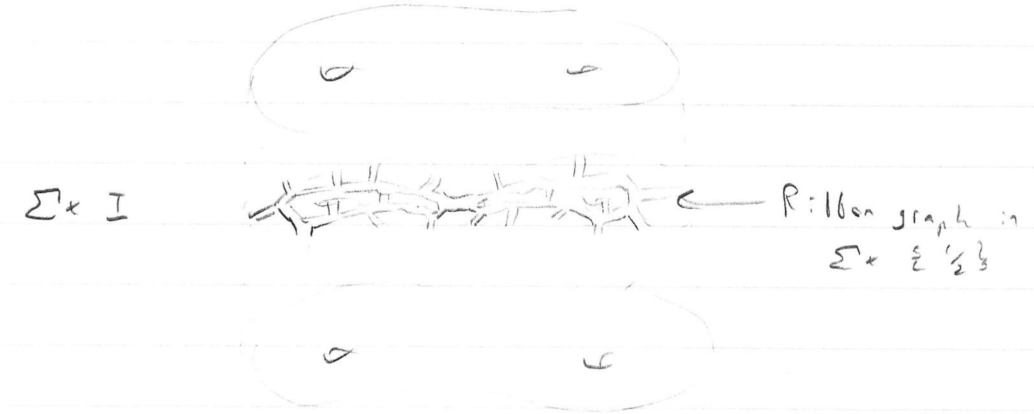
comult & mult
counit & unit <u>adjoint</u>.

<u>Example</u>

Frobenius algebras in $Rep(LSU(2)_k)$ are naturally classified by the ADE Dynkin diagrams

There is a construction that takes as input a chiral CFT with Frobenius structure, & outputs a full CFT. We'll at least describe how to compute the partition function of the result.

Picture



Pieces of ribbon are labelled by the Frobenius algebra object, & the (directed) trivalent junctions are labelled by the multiplication & comultiplication.

Recall, a closed Σ must map to a functor $Vect \longrightarrow Vect$, which has form $X \longmapsto X \otimes V$. We call V the space of conformal blocks

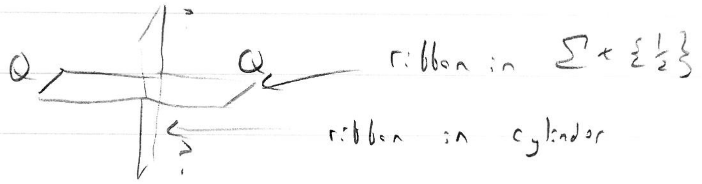
Furthermore, there is a canonical element ω in V: the image of $\lambda \in \mathbb{C}$ under $\mathbb{C} \longrightarrow V$ (the other piece of data).

Now, there is a gadget: a 3d TQFT which assigns to $\Sigma \times I$ an element of the conformal blocks of $\partial(\Sigma \times I) = \Sigma \times \overline{\Sigma}$ ← orientation reverse, i.e. an element of $V \otimes V^*$. Call it c.

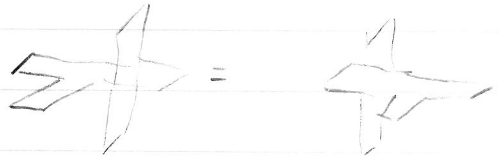
Then put $Z(\Sigma) = \langle c, \omega \otimes \omega^* \rangle$.

How do we produce the 3-d TQFT?

Take a curve $C \subseteq \Sigma$, & look at the cylinder $C \times I \subset \Sigma \times I$. Put a ribbon structure on this cylinder. We introduce labels on these new ribbons



Satisfying some compatibilities, e.g



Choose the universal assignment such that these are satisfied. Produce the "<u>full centre</u>"

$$Z_{full}(Q) = \bigoplus_{\lambda,\mu} \mathrm{Hom}_{Q,Q}(\lambda \boxtimes^{+} Q \boxtimes^{-} \mu^{\vee}, Q) \otimes \underbrace{\lambda \otimes \mu}_{\in\, \mathcal{C} \otimes \bar{\mathcal{C}}}$$

The <u>state space</u> of the full CFT associated to our chiral CFT & Frobenius algebra object Q is then

Fuchs-Runkel-Schweigert

$$H_{full} = \bigoplus_{\lambda,\mu} \mathrm{Hom}_{Q,Q}(\lambda \boxtimes^{+} Q \boxtimes^{-} \mu^{\vee}, Q) \otimes H_\lambda \otimes \bar{H}_\mu .$$

$\boxtimes^{\pm}$ is "tensor from above/below" via braided monoidal structure. $\mathrm{Hom}_{Q,Q}$ means as bimodules, $\lambda, \mu \in \mathcal{C}$.

This is what the full CFT assigns to a <u>circle</u>.

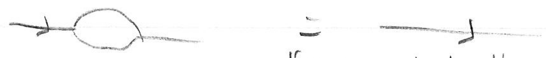
<u>Definition</u>

Write $A \simeq_M B$ if A, B are <u>Morita equivalent</u>, i.e.

$\exists\ {}_A X_B,\ {}_B Y_A$ s.t. ${}_A X \otimes_B Y_A \cong {}_A A_A$, ${}_B Y \otimes_A X_B \cong {}_B B_B$.

Recall:

In our Frobenius algebra, we have relation



So we can "Fill holes" in our ribbon graph

Our picture becomes $\Sigma \times [0,1]$ with an embedded surface. Decorated by a defect, or surface operator. This is the label of the Frobenius algebra object Q, roughly.

Conformal Nets

Definition

A conformal net is a cts functor

$$\mathcal{A} : \left\{ \begin{array}{c} \text{contractible 1-} \\ \text{manifolds} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{von Neumann} \\ \text{objects} \end{array} \right\}$$

with embeddings   with homs & anti homs

Such that

- $\mathcal{A}[0,1]$ & $\mathcal{A}[1,2]$ say commute in, & generate a dense subalgebra of, $\mathcal{A}[0,2]$.
- $\mathcal{A}[0,1] \otimes \mathcal{A}[2,3] \longrightarrow \mathcal{A}[0,3]$
  $\cap$
  $\mathcal{A}[0,1] \boxtimes \mathcal{A}[2,3]$ lifts

  $\otimes$?
- $\{\phi \in \mathrm{Diff}[0,1] : \text{identity near boundary}\} \longrightarrow \mathrm{Aut}(\mathcal{A}[0,1])$ lands in inner auts
- another, complicated axiom.

Somehow these should really correspond to 3d TQFTs. To get 2d chiral CFTs, one might impose a further "positive energy" condition.

Want to associate a von Neumann algebra to a point $x \in \Sigma$. Look at interval $\{x\} \times I \subseteq \Sigma \times I$.

If $Q = 1$ "cardy case", there is no defect, so take von Neumann algebra $\mathcal{A}(\{x\} \times I)$.

Definition:

A bicoloured interval is a contractible 1-manifold equipped with a decomposition $I = I_{red} \cup I_{blue}$ into connected submanifolds, together with a local co-ord at the colour-changing point

Let $\mathcal{A}, \mathcal{B}$ be conformal nets.

Definition

A defect between $\mathcal{A}$ & $\mathcal{B}$ is a function

$$D : \left\{ \begin{matrix} \text{bicoloured} \\ \text{intervals} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{von Neumann} \\ \text{algebras} \end{matrix} \right\}$$

morphisms $\underset{\substack{\text{colour preserving} \\ \text{embedding}}}{I \hookrightarrow J}$ as before

such that $D(I) = \mathcal{A}(I)$ if $I_{blue} = \emptyset$

& $D(I) = \mathcal{B}(I)$ if $I_{red} = \emptyset$,

& axioms like those for conformal nets.

Examples of Frobenius Algebra Objects

Example 0: The unit object $\mathbb{1} \in \mathcal{C}$.

Example 1: For any object $X \in \mathcal{C}$, then $Q := X \boxtimes X^\vee$ is the "matrix algebra" on X, e.g. if $\mathcal{C} = \mathrm{Rep}(LSU(2)_2)$, $X = V_1$, we have $Q = V_0 \oplus V_2$. Morita equivalent to $\mathbb{1}$.

Example 2: In $\mathcal{C} = \mathrm{Rep}(LSU(2)_k)$, we have

$$V_0 \boxtimes V_0 = V_0$$

$$V_0 \boxtimes V_k = V_k = V_k \boxtimes V_0$$

$$V_k \boxtimes V_k = V_0. \qquad \text{closed under } \boxtimes$$

So $Q = V_0 \oplus V_k$ by analogy? This only sometimes works: only if k is even. If k is odd, the associator is twisted by a non-trivial 3-cocycle.

Now, recall we're trying to produce an extended CFT from a conformal net $\mathcal{A}$ & a Frobenius algebra object Q. Recall also the monoidal structure on $\mathrm{Rep}(\mathcal{A})$. Let $A := \mathcal{A}([0,1])$. Given a representation H of $\mathcal{A}$, we pick a diffeomorphism between the two half circles reversing orientation, & thus producing a fully faithful embedding

$$\mathrm{Rep}(\mathcal{A}) \hookrightarrow A, A \text{ bimodules}$$

& we apply Connes fusion in this setting and check we come from $\mathrm{Rep}(\mathcal{A})$.

The algebra associated to a point

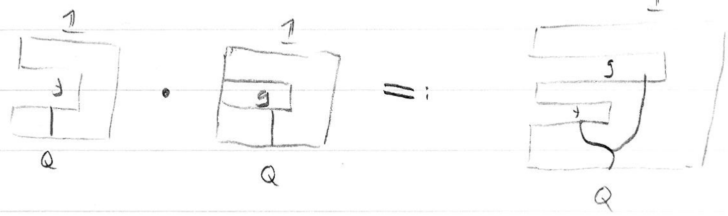
$B := \mathrm{Hom}(L^2A_A, Q_A)$. This admits a von Neumann algebra structure.

Abbreviate L^2A to $\mathbb{1}$, and $\boxtimes_A$ to $\boxtimes$.

Product: Given $f, g \in B$, we can compose

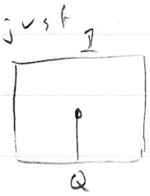
$$\mathbb{1} \xrightarrow{g} Q \cong \mathbb{1} \boxtimes Q \xrightarrow{f \otimes 1} Q \boxtimes Q \xrightarrow{m} Q,$$

or graphically



to define the composite.

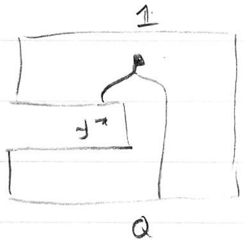
Unit: $\eta : \mathbb{1} \longrightarrow Q$ the unit of Q.
The picture is just



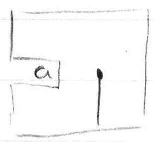
Star: If $f \in B$, we need to define f^*. Use

$$\mathbb{1} \xrightarrow{\eta} Q \xrightarrow{\Delta} Q \boxtimes Q \xrightarrow{f^* \otimes 1} \mathbb{1} \boxtimes Q \xrightarrow{\sim} Q$$

or graphically



Embedding $A \hookrightarrow B$: Define

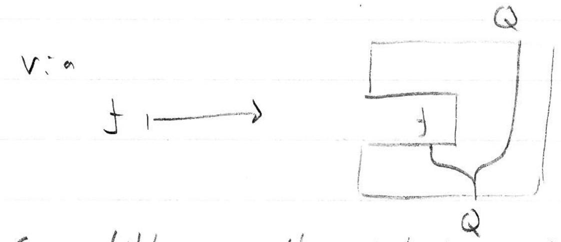
$a \longmapsto$ 

viewing a as acting on $L^2 A$.

One checks this is a $*$-algebra, using all the Frobenius axioms for Q.

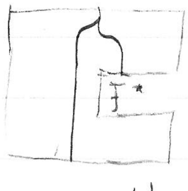
Action of B on Q: $B \longrightarrow B(Q)$

via



Compatibly with product using associativity of m.

Q is actually also a right B-module, with right action by taking F^* left A-linear:



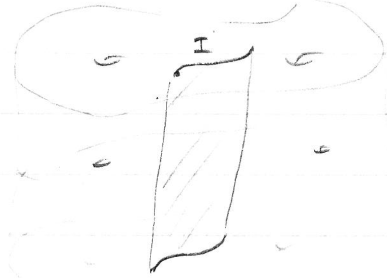
One can reinterpret this construction as the construction of a defect D, such that

$$D([0,1]) = B$$

whenever $[0,1]$ has both colours. It has the special property that the location of the colour change doesn't matter. We call it a topological defect.

The bimodule associated to an interval:

Recall the Kapustin - Saulina idea: we replace Σ by $\Sigma \times [0,1]$ with a defect at $\Sigma \times \{\frac{1}{2}\}$. we replaced pt by $[0,1]$ above. Here the picture is



Given interval I, consider $I \times [0,1]$. Only focus on $\partial(I \times [0,1])$. We have collars at the end of our interval that allow us to smooth out this rectangle to a circle.

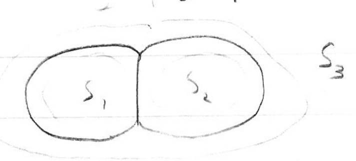
We'll need a coordinate independant way of studying $\mathrm{Rep}(\mathcal{A})$.

Given an abstract circle S, formulate

$$\mathrm{Rep}_S(\mathcal{A}) := \left\{ \begin{array}{l} \text{Hilbert spaces with actions} \\ \text{of } \mathcal{A}(I),\ \forall I \subsetneq S \end{array} \right\}$$

a category equivalent to $\mathrm{Rep}(\mathcal{A})$, but not canonically (would need to pick a diffeo from S to the standard circle).

Draw a trivalent graph with 3 embedded circles as follows:



all assumed to be equipped with compatible smooth structures. Then there is a canonical product

$$\mathrm{Rep}_{S_1}(\mathcal{A}) \times \mathrm{Rep}_{S_2}(\mathcal{A}) \longrightarrow \mathrm{Rep}_{S_3}(\mathcal{A})$$

There's a functor

S' standard circle.

$$\begin{array}{ccc} \mathrm{Rep}(\mathcal{A}) & \xrightarrow{F_S} & \mathrm{Rep}_S(\mathcal{A}) \quad \text{by} \\ H & \longmapsto & H \times_{\mathrm{Diff}(S')} \mathrm{Diff}(S', S) \end{array}$$

$\mathrm{Diff}(S')$ acts on H by the axioms of conformal nets: that certain local diffeos associated to $\mathcal{A}$ are inner. The action is only a projective action from ambiguity of inner automorphism representative choices.

Want to say: the value of the full CFT on I is the image $Q[I]$ of Q under the functor F_S, for S being $\partial(I \times [0,1])$.

This doesn't quite work as written, however our S is equipped with an involution by swapping the copies of I: reflection. S^1 also has such an involution. Write Diff^{sym} for diffeos compatible with this. So define

$$\begin{aligned} \mathrm{Rep}(\mathcal{A}) &\longrightarrow \mathrm{Rep}_S(\mathcal{A}) \\ H &\longmapsto H \times_{\mathrm{Diff}^{sym}(S^1)} \mathrm{Diff}^{sym}(S^1, S) \end{aligned}$$

This now works. Furthermore, we have two copies of $\mathcal{A}([0,1]) = A$ acting, by the two inclusions $[0,1] \hookrightarrow \partial(I \times [0,1])$.

These actions extend in a canonical way to actions of B.

Given $Q[I_1]$, $Q[I_2]$, one can show there's a canonical isomorphism of B,B bimodules

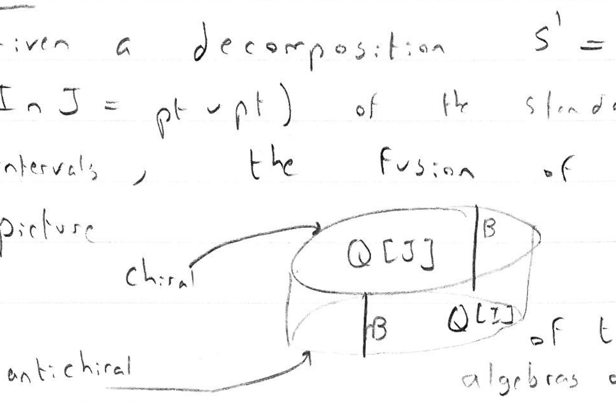
$$Q[I_1] \boxtimes_B Q[I_2] \longrightarrow Q[I_1 \cup I_2].$$

That's not two hard. What one wants to check is that we recover the state space from before by gluing up a circle. Namely, we expect to recover

$$\mathcal{H}_{Full} = \bigoplus_{\lambda,\mu} \mathrm{Hom}_{QQ}(\lambda \boxtimes^{+} Q \boxtimes^{-} \mu, Q) \otimes H_\lambda \otimes \overline{H_\mu}$$

<u>Theorem</u>:

Given a decomposition $S^1 = I \cup J$ (where $I \cap J = pt \cup pt$) of the standard circle into intervals, the fusion of bimodules from picture



is canonically iso to $\mathcal{H}_{Full}$ as reps of the chiral & antichiral algebras on the circles.

Proof sketch:

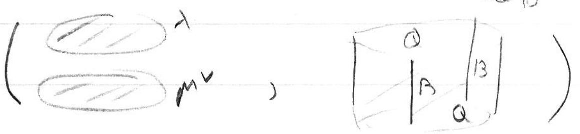
Use **Lemma**: For an A-module H,

H is a Q-module $\iff$ H is a B-module.

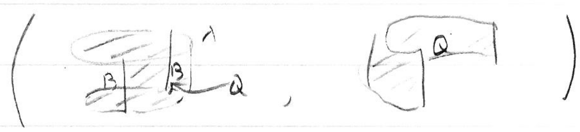
where Q-module means Q-module object.

This allows us to proceed.

$$\mathrm{Hom}(H_\lambda \otimes \bar{H}_\mu , Q[I] \boxtimes_{B \hat{\otimes} B^{op}} Q[J])$$

$= \mathrm{Hom}$ 

(pictorially)

$= \mathrm{Hom}$ 

by duality

$= \mathrm{Hom}_{B,B}$ 

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□