# Dualizability in Low dimensional Higher Category Theory

What are higher categories?

objects, morphisms, 2-morphisms, ...

2-morphisms can be composed in a variety of ways: horizontally or vertically.

Example: Bicategory or weak 2-category.

Examples:

1. Monoidal category $(\mathcal{C}, \otimes)$ yields a 2-category with one object: pt, morphisms the objects of $\mathcal{C}$, & 2-morphisms the morphisms of $\mathcal{C}$. Call this 2-category $B(\mathcal{C}, \otimes)$. Can compose vertically by usual composition, & horizontally by $\otimes$.
2. Cat is naturally a 2-category.
3. Morita category of algebras, bimodules, maps.
4. Fundamental $n$-groupoid of a space
   - objects: points
   - morphisms: paths
   - 2-morphisms: paths between paths
   - ⋮
   - $n$-morphisms: paths between ... between paths up to htpy.

## Baez-Dolan Hypotheses for higher cats

Stabilization Hypothesis: (Periodic table)

| $n=-2$ | $n=-1$ | $n=0$ | $n=1$ | $n=2$ | |
|---|---|---|---|---|---|
| $\{*\}$ | $\{*, \emptyset\}$ | set | category | 2-category | $k=0$ |
| " | $\{*\}$ | Monoid | Monoidal category | Monoidal 2-category | $k=1$ |
| " | " | Commutative monoid | braided monoidal category | braided monoidal 2-category | $k=2$ |
| " | " | " | Symmetric monoidal category | Sylleptic monoidal 2-category | $k=3$ |
| " | " | " | " | Symmetric monoidal 2-category | $k=4$ |

Encapsulates $k+n$ - categories with $k$ trivial layers: 1 object, 1 morphism, ..., 1 $k$-1-morphism.

<u>Example</u>:

A 2-category with 1 object & 1-morphism is a set of 2-morphisms with two binary operations, satisfying a certain distributivity.

Eckmann-Hilton $\Rightarrow$ the operations agree & are commutative.

The <u>stabilisation hypothesis</u> says that the forgetful functor from $(k+1)$-monoidal $n$-categories to $k$-monoidal $n$-categories is an <u>equivalence</u> if $k \geq n+2$.

<u>Homotopy Hypothesis</u>:

If $X$ is a space, theres a functor

$$\begin{aligned} \Pi_1 : \text{Spaces} &\longrightarrow \text{Gpd} \\ X &\longmapsto \Pi_1(X). \end{aligned}$$

One can go back via

$$\begin{aligned} \text{Gpd} &\longrightarrow \text{Spaces} \\ A &\longmapsto |NA| \end{aligned}$$

Composing, $|\Pi_1 X|$ is a <u>1-type</u> for $X$, i.e., has the same fundamental groupoid, and $\pi_k(|\Pi_1 X|) = 0$ if $k > 1$.

The <u>homotopy hypothesis</u> says, the homotopy theory of $n$-groupoids should be equivalent to the homotopy theory of $n$-types.

If $n \longrightarrow \infty$, this says $\infty$-groupoids are in some sense spaces (homotopy theoretically).

"Definition":

An $(\infty, n)$ - category is an $\infty$ - category where all morphisms are invertible above $n$.

There are several ways of making this hypothesis precise. Model cats do it.

Theorem (Barwick - SP)

4 axioms characterise the homotopy theory of $(\infty, n)$ - categories up to equivalence.

We'll only talk about one model.

By the way, we'll discuss the 3rd hypothesis - the Cobordism hypothesis - later.

Segal n - categories

Denote by $\Delta$ the category of combinatorial simplices, i.e finite sets $[n] = \{0, \dots, n\}$ with order preserving maps. View $\Delta$ as a subcategory of Cat, via $[n] \longmapsto$ the category with $n+1$ objects, & 1 morphism $i \to j$ if $i \leq j$.

For each cat $X$, one can construct its nerve $NX$, which is a presheaf of Sets on $\Delta$,

$$NX : \Delta^{op} \longrightarrow Set$$

via $N(X)[i] = Fun([i], X)$

so $N(X)[1] \cong \coprod_{a,b} X(a,b)$

$N(X)[2] \cong \coprod_{a,b,c} X(a,b) \times X(b,c)$ etc.

Tells us everything about $X$, as 3 maps $[1] \rightrightarrows [2]$ give

$$NX[1] \times_{NX[0]} NX[1] \xleftarrow{\sim} NX[2] \longrightarrow NX[1]$$

pairs of composable morphisms

composition.

## Definition

A Segal category has a set of objects $X_0$, with for each pair of objects a space $X(a,b)$ which we assemble to get $X_1 = \coprod_{a,b} X(a,b)$. Similarly, require a space $X_n = \coprod_{a_1,\dots,a_n} X(a_1,\dots,a_n)$, such that $X_\bullet$ is a simplicial space satisfying some conditions. Have maps

$$X(a,b) \times X(b,c) \xleftarrow{\phi} X(a,b,c) \longrightarrow X(a,c)$$

& require $\phi$ to be a htpy equivalence. Similarly $X(a_0,\dots,a_n) \longrightarrow X(a_0,a_1) \times \dots \times X(a_{n-1},a_n)$.

These infinitely many conditions are called the Segal conditions.

Work on this is due to Berger, Tamsamani, Hirschowitz - Simpson, Pellissier, ...

## Theorem:

There is a Quillen model structure on simplicial spaces with $X_0$ discrete, where the cofibrations are monomorphisms, & fibrant objects are Segal cats (satisfying conditions: Reedy fibrant). Fibrations have RLP for $pt \hookrightarrow NJ$.

RLP = right lifting property

J = "free walk iso" gpd $\bullet \rightleftarrows \bullet$

Higher version is similar

A Segal n-category has a set $X_0$ of objects, Segal $(n-1)$-cats of k-morphisms $X_k$, forming a simplicial Segal $(n-1)$-category.

Require also a segal condition. But we need to know what weak equivalences are in Segal $(n-1)$.

Just for Segal cats, if $f: X \to Y$ is a map of Segal cats, put $hX$ the cat with objects $X_0$, $\hom(a,b) = \pi_0 X(a,b)$.

So get

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ hX \arrow[r, "hf"] & hY \end{tikzcd}$$

$f$ is an equivalence if

1) $hf$ is an equiv of cats

2) get $hf_{()}$ equivs $X(a,b) \to Y(fa, fb)$.

One repeats to do for Segal $n$-cats.

# Duals in 2-Categories

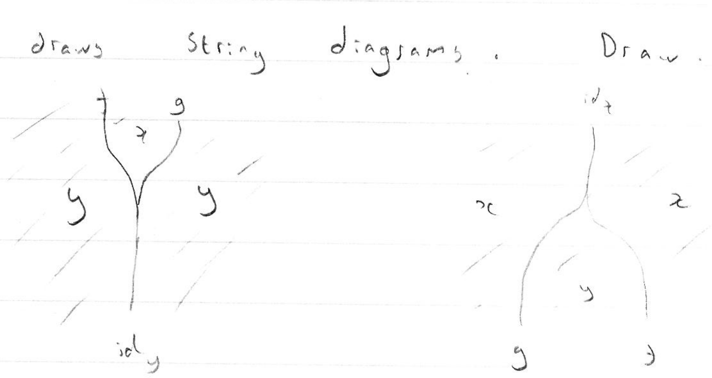
**Definition:**

A 1-morphism $f: x \to y$ is (left) dualisable if there is a $g: y \to x$, and $ev: fg \to 1_y$, $coev: 1_x \to gf$ satisfying

$$\left( f \xrightarrow{1 \cdot coev} fgf \xrightarrow{ev \cdot 1} f \right) = 1_f$$

& $$\left( g \xrightarrow{coev \cdot 1} gfg \xrightarrow{1 \cdot ev} g \right) = 1_g .$$

One draws string diagrams. Draw.




& the equations become


**Example:**

From a monoidal category, e.g. $(Vect_k, \otimes)$, form 2-category BVect. When is a vector space $X$ dualisable? If there is an $X^*$, and

$ev: X \otimes X^* \to k$.

If one also has $coev: k \to X \otimes X^*$, then $X$ must be finite-dimensional.

**Example:**

Consider the 2-category Cat. A functor is left dualisable if it is a left adjoint.

ev & coev are the unit & counit.

## Advantages of Segal $n$-Categories

- Firstly, it's relatively easy to construct examples.
- It's possible to compute functor Segal $n$-categories.
- It's easy to extract pieces of a higher category. We saw we could extract the $X(a,b)$ : Segal $(n-1)$-categories. We also could find $hX$.

From a Segal $n$-category $X$, we can extract various 2-categories, e.g. objects, 1-morphisms, 2-morphisms up to equivalence. From this, we can ask about dualisability for 1-morphisms.

Alternatively, can get a 2-category with, say $(k-2)$-, $(k-1)$- & $k$-morphisms, giving a theory of duals for $(k-1)$-morphisms.

## Approaches to Symmetric Monoidal Structures

- algebras for an $E_\infty$-operad
- $\Gamma$-objects
- Use Stabilisation hypothesis.

All we really care about is that $hX$ is symmetric monoidal. So left & right duals agree for objects.

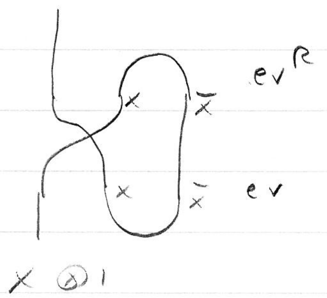
## Definition

If $(\mathcal{C}, \otimes)$ is a symmetric monoidal $(\infty, n)$-category, say it is $k$ fully-dualisable if objects, morphisms, ..., $(k-1)$-morphisms all have both left & right duals.

**Example** of extra structure with duals.

Suppose $(\mathcal{C}, \otimes)$ is 2-fully dualisable.

For any object $X$, we have a canonical "serre automorphism" $X \longrightarrow X$.




$$\begin{array}{cl} X \otimes 1 & \\ \downarrow & ev^R \\ X \otimes X \otimes \bar{X} & \\ \downarrow & \text{swap } 1^{st} \text{ two factors} \\ X \otimes X \otimes \bar{X} & \\ \downarrow & ev \\ X \otimes 1 & \end{array}$$

$ev^R$ is the right dual to the evaluation morphism.

We'll discuss some properties of bordism categories

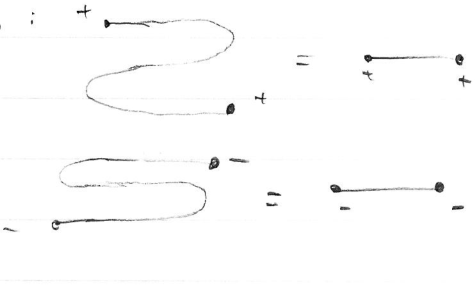
**Theorem** (arXiv: 1112.1000)

$(\mathrm{Bord}_1^{or}, \amalg)$ is presented as a sym monoidal category by

generating objects: $\bullet_+$ , $\bullet_-$

generating morphisms:


relations:




**Corollary**

Symmetric monoidal functors satisfy

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{or}, \mathcal{C}) \cong \{(\underset{+}{X}, \underset{-}{\bar{X}}, ev, coev)\}$$

exercise 1.2 $\rightarrow$ $\cong k(\mathcal{C}^{fd})$

fully dualisable objects

maximal groupoid.

This is a form of the <u>cobordism hypothesis</u>.

<u>Cobordism Hypothesis:</u>

Roughly, this says $\mathrm{Bord}_n^{\text{framed}}$ is the free symmetric monoidal $(\infty, n)$-category generated by a single $n$-fully dualisable objects

Proven for $(\infty, 2)$-cats by Hopkins-Lurie
$(\infty, n)$-cats by Lurie.

In a precise sense, $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \simeq \underbrace{k(\overbrace{\mathcal{C}^{fd}}^{n\text{-fully dualisable objects}})}_{\text{maximal } \infty\text{-groupoid}}$

Now, whenever one has a mapping object $\mathrm{Maps}(\mathcal{B}, \mathcal{C})$, one has an action of $\mathrm{Aut}(\mathcal{B})$.
So as $O(n)$ acts on $\mathrm{Bord}_n^{fr}$, we get a (coherent homotopy) action of $O(n)$ on $k(\mathcal{C}^{fr})$.

If $G \longrightarrow O(n)$, we can ask for lifts

$$\begin{tikzcd} & BG \arrow[d] \\ M \arrow[r, "\tau"] \arrow[ur, dashed] & BO(n) \end{tikzcd}$$

for a manifold $M$.

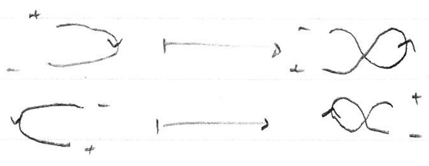
We also have

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{G}, \mathcal{C}) \simeq \underbrace{[k(\mathcal{C}^{fd})]^{hG}}_{\text{htpy fixed points}}$$

What about our $n=1$ example. Should have an action of $\mathbb{Z}/2$. or a non-trivial $\mathrm{Bord}_1^{or} \to \mathrm{Bord}_1^{or}$.
Indeed, put

$+ \longmapsto -$

$- \longmapsto +$



Should expect $(\mathrm{Bord}_n^G) = (\mathrm{Bord}_n^{fr})_{/hG}$.

In this example

$$(\mathrm{Bord}_1^{or})_{/h(\mathbb{Z}/2)} \overset{?}{=} \mathrm{Bord}_1^{O(1)}$$

$$= \mathrm{Bord}_1^{unoriented}$$

Theorem:

$\mathrm{Bord}_1^{un}$ has the presentation

- generating object:
- generating morphisms:
- relations:

How do we take the homotopy quotient for spaces? If $X \curvearrowleft G$, replace $X$ by $X \times EG$, & quotient there. We want to do this for higher categories.

What plays the role of $E\mathbb{Z}/2$? Use the category $J$, with objects $\{j, j^{-1}\}$

isomorphisms $\{1_j, 1_{j^{-1}}, j \rightleftarrows j^{-1}\}$

$J$ admits a free $\mathbb{Z}/2$-action.

Definition

If $(\mathcal{C}, \otimes)$ is symmetric monoidal, there is a symmetric monoidal category $\mathcal{C} \boxtimes J$ s.t.

$$\mathrm{Fun}^{\otimes}(\mathcal{C} \boxtimes J, \mathcal{D}) = \mathrm{Fun}^{\otimes}(J, \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}))$$

So $(\mathrm{Bord}_1^{or} \boxtimes J)$ has a presentation

objects: pairs $(b, j)$

morphisms: $(b \to b', 1_j)$, $(1_b, j \to j')$

$(b, j) \otimes (b', j) \xrightarrow{\sim} (b \otimes b', j)$

relations: those in $\mathrm{Bord}_1^{or}$, those in $J$, $(b \to b', 1_j)$, $(1_b, j \to j')$ commute, &

$$\begin{tikzcd}
(b, j) \otimes (b', j) \arrow[r, "\sim"] \arrow[d] & (b \otimes b', j) \arrow[d] \\
(b', j) \otimes (b, j) \arrow[r, "\sim"] & (b' \otimes b, j)
\end{tikzcd}$$

commutes.

Now we can take the homotopy quotient, by quotienting by the free $\mathbb{Z}/2$ action

$$\left(\mathrm{Bord}_1^{or} \otimes J\right) / \mathbb{Z}/_2 = \left(\mathrm{Bord}_1^{or}\right)_{h\mathbb{Z}/2}.$$

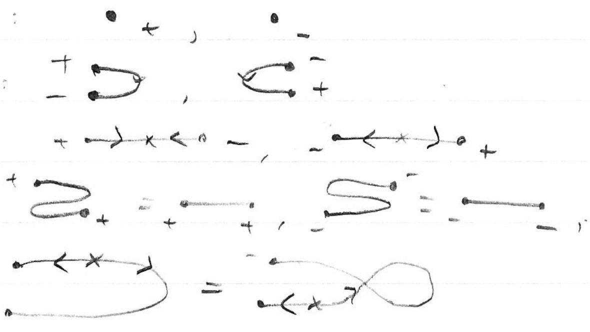
With the presentation we already have, we get $\left(\mathrm{Bord}_1^{or}\right)_{h\mathbb{Z}/2}$ has

generating objects:

generating morphisms:

Via $[x, j] \mapsto [x, j^{-1}]$

relations:



This gives our functor to the unoriented bordism category. Exercise 2.3 allows us to verify it is an equivalence, as required.

In today's talk, we'll increase the level of dualisability. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal 2-cat which is 2-fully dualisable.

So in this case, the cobordism hypothesis says
$$k(\mathcal{C}^{fd}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{fr}, \mathcal{C}).$$
$O(2)$ acts on the right, so homotopy acts on the left. We'll understand this action. Note $SO(2) \ltimes \mathbb{Z}/_2 \cong O(2)$. We already got a grip on the $\mathbb{Z}/_2$ action, so we'll focus on $SO(2)$.

We want a map $SO(2) \longrightarrow \mathrm{Aut}(k(\mathcal{C}^{fd}))$.
In other words, for each point in $SO(2)$ we should get a functor $k(\mathcal{C}^{fd}) \longrightarrow k(\mathcal{C}^{fd})$. For each path we should get a natural isomorphism, & for higher paths we should get higher transformations.

$SO(2)$ is connected, so we land in the identity functor component $\mathrm{Aut}^{\circ}(k(\mathcal{C}^{fd}))$.

We also saw last time, in this setting we have Serre automorphisms of objects.

Homework 1.2, 3.4 show this is a natural transformation $S: \mathrm{Id}_{k(\mathcal{C}^{fd})} \longrightarrow \mathrm{Id}_{k(\mathcal{C}^{fd})}$.

Now, $\pi_1(SO(2)) = \mathbb{Z}$. We expect the Serre automorphism to come from a non-trivial element in $\pi_1(SO(2))$.

Consider maps $BSO(2) \longrightarrow B\mathrm{Aut}^{\circ}(k(\mathcal{C}^{fd}))$

Have a filtration

$$\begin{tikzcd}
BSO(2) \arrow[r] & BAut^{\circ}(k(\mathcal{C}^{fd})) \\
\mathbb{CP}^{\infty} \arrow[u, equal] & \\
\cup\!\mathsf{I} & \\
\vdots & \\
\cup\!\mathsf{I} & \\
\mathbb{CP}^2 & \\
\cup\!\mathsf{I} & \\
S^2 \arrow[uuuuuuur, bend right] &
\end{tikzcd}$$

^This will be the Serre automorphism.

For today, let $\mathcal{C}$ be a 2-category. Then $k(\mathcal{C}^{fd})$ is a 2-type, so $Aut(k(\mathcal{C}^{fd}))$ is also a 2-type & $BAut(k(\mathcal{C}^{fd}))$ is a 3-type.

Note all spaces $\mathbb{CP}^2 \subseteq \mathbb{CP}^3 \subseteq \cdots \subseteq \mathbb{CP}^{\infty}$ have the same 3-type, so suffices to lift out map from $S^2$ to $\mathbb{CP}^2$.

In fact, $BAut^{\circ}(k(\mathcal{C}^{fd}))$ is a <u>simply-connected 3-type</u>.

## <u>Whitehead's Certain Exact Sequence</u>

Given a space $X$, we can look at

$$X \longrightarrow Sym^{\infty} X.$$

<u>pointed</u>

Say $X$ is a simply connected 3-type. Let $F \to X$ be the homotopy fibre of this map, & look at the long exact sequence of homotopy

<u>Dold-Thom Theorem</u> says $\pi_k(Sym^{\infty} X) = H_k(X)$

Alternatively, we could use a configuration space of points in $X$, labelled by elts of $\mathbb{Z}$, topologised such that points can disappear into base point, & when points come together, the labels add. Have instead, fibre sequence

$$F \longrightarrow X \longrightarrow Conf(X, \mathbb{Z})$$

$Sym^\infty$ is essentially the same, with $\mathbb{Z}$ replaced by $\mathbb{N}$. Again,

$$\pi_k(\mathrm{Conf}(X, \mathbb{Z})) = H_k(X),$$

& we can look at l.e.s of the fibre sequence.

<u>Hurewicz Theorem</u> $\Rightarrow$ $\pi_2 X \cong H_2 X$

$\pi_3 X \to H_3 X$ surjective

So we get the l.e.s

$$0 \longrightarrow H_4 X \longrightarrow \Gamma X \xrightarrow{q} \pi_3 X \longrightarrow H_3 X \longrightarrow 0$$

<u>Exercise</u> $\Gamma X = \Gamma \pi_2$.

<u>Theorem</u> (Whitehead)

This exact sequence is a complete invariant of simply-connected 3-types. (Possibly for fixed $\pi_2$).

You could also look at the Postnikov tower

$$\begin{array}{ccc} K(\pi_3, 3) \longrightarrow X & & \\ \downarrow & & \\ K(\pi_2, 2) & \xrightarrow{q} & K(\pi_3, 4) \end{array}$$

classified by $q \in H^4(K(\pi_2, 2); \pi_3)$.

Again, by Hurewicz, $H_3(K(\pi_2, 2)) = 0$, & so by universal coefficients,

$$H^4(K(\pi_2, 2); 3) \cong \mathrm{Hom}(H_4(K(\pi_2, 2)), \pi_3).$$

So

$$\Gamma X = H_4(K(\pi_2, 2)),$$ & the two $q$'s agree.

## Whitehead's $\Gamma$-functor

$\Gamma$ is a functor $Ab \longrightarrow Ab$ : $\Gamma(A)$ is generated by symbols $\gamma(a)$, $a \in A$, with relations

1) $\gamma(a) = \gamma(-a)$
11) $\gamma(a) + \gamma(b) + \gamma(c) + \gamma(a+b+c)$
$= \gamma(a+b) + \gamma(a+c) + \gamma(b+c)$.

$A \mapsto \Gamma(A)$ is the universal quadratic map, where quadratic means satisfying 1) & 11).

So $\mathrm{Hom}(\Gamma(A), B) = \mathrm{Hom}_{\text{quadratic}}(A, B)$

Thus simply-connected 3-types are classified by

- $\pi_2$
- $\pi_3$
- A quadratic map $\pi_2 \longrightarrow \pi_3$

Exercise: Compute that $\Gamma(\mathbb{Z}/n) \cong \mathbb{Z}/n$ if $n$ is odd
$\Gamma(\mathbb{Z}/2) \cong \mathbb{Z}/4$.

So we have correspondences

$$\left\{\begin{array}{c}\text{pointed simply-}\\ \text{connected 3-types}\end{array}\right\} \iff \left\{\begin{array}{c}\text{pointed simply-}\\ \text{connected 3-groupoids}\end{array}\right\} \iff \left\{\begin{array}{c}\text{braided monoidal}\\ \text{Cat with invertible}\\ \text{objects \& morphisms}\end{array}\right\}$$

homotopy hypothesis

Stabilisation hypothesis

$$\Updownarrow \qquad\qquad\qquad\qquad \|\ \text{def}$$

$$\left\{(\pi_2, \pi_3, \pi_2 \xrightarrow{\text{quadratic}} \pi_3)\right\} \qquad\qquad \left\{\text{braided 2-groups}\right\}$$

There's a good reference for this:
Joyal & Street - "Braided monoidal categories"
Look for the unpublished version.

We can extract the $\pi_2, \pi_3, \pi_2 \longrightarrow \pi_3$ data from a braided 2-group:

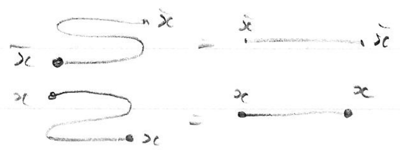
Let $X$ be a braided 2-group.

- $\pi_2$ is the group of iso classes of objects.
- $\pi_3 = \mathrm{Aut}(\mathbb{1})$, $\mathbb{1} \in X$ unit object.

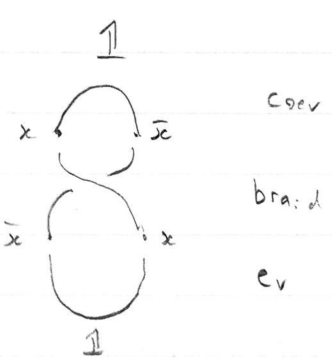
What about the quadratic map? Given $x \in X$, there exist an "inverse" $\bar{x}$, i.e. there are pairings $\bar{x} \otimes x \xrightarrow[\sim]{ev} \mathbb{1}$

$$\mathbb{1} \xrightarrow[\sim]{coev} x \otimes \bar{x}$$

satisfying the usual equations, i.e.




Take $x \in X$. We'll build an automorphism $\mathbb{1} \to \mathbb{1}$ via:




Exercise: Use problem 1.2 to show this is well-defined, & only depends on $x$ up to isomorphism. Also show it's quadratic in $x$.

Finally, let's go back to the lifting problem.

| | $\mathbb{CP}^2$ | $S^2$ |
|---|---|---|
| $\pi_3$ | $0$ | $\mathbb{Z}$ |
| $\pi_2$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $q$ | $0$ | $\mathbb{Z} \to \mathbb{Z}$<br>$n \mapsto n^2$ |

$\nearrow$

$\mathbb{Z} = \Gamma(\mathbb{Z})$

We have $S^2 \xrightarrow{\text{Serre}} B\mathrm{Aut}^0(k\mathrm{l}e^{fd}))$

Look at Hopf map $S^3 \xrightarrow{h} S^2$, & consider

$$\begin{tikzcd} S^3 \arrow[r, "h"] \arrow[d] & S^2 \arrow[r, "\text{Serre}"] \arrow[d] & BAut^\circ(k(\mathcal{C}^{fd})) \\ D^4 \arrow[r] & \mathbb{CP}^2 \arrow[ur, dashed] & \end{tikzcd}$$

lift via this diagram.

$Aut^\circ(k(\mathcal{C}^{fd}))$ is a simply connected 3-type

$\pi_2$ = natural auts of $id_{k(\mathcal{C}^{fd})}$

$\pi_3$ = natural auts of $id_{id_{k(\mathcal{C}^{fd})}}$

$q : \pi_2 \longrightarrow \pi_3$ constructed as follows:

Fill in squares

$$\begin{tikzcd} & x \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow s_f"] & \\ x \arrow[ur, "s_x"] \arrow[dr, "f"'] & & y \\ & y \arrow[ur, "s_y"'] & \end{tikzcd}$$

for any $f$.

Exercise 3.4 let's us see how to do this.

Then specialise to the case where $f = s_x^{-1} : x \longrightarrow x$.

one has

$$\begin{tikzcd} & \Downarrow \text{coev} & \\ & x \arrow[dr, "s_x^{-1}"] \arrow[dd, phantom, "\Downarrow s_f"] & \\ x \arrow[ur, "s_x"] \arrow[dr, "s_x^{-1}"'] & & x \\ & x \arrow[ur, "s_x"'] & \\ & \Downarrow \text{ev} & \end{tikzcd}$$

is a natural transformation $q(s) \in Aut(id_{id})$

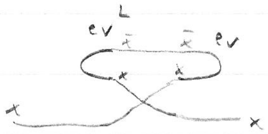
## Ultimate Challenge Exercise:

Use naturality from 3.4 to calculate this, & show $q(s)$ is trivial.

Joint work with Chris Douglas & Noah Snyder: increasing dualisability to <u>three</u> layers.

Let $(\mathcal{C}, \otimes)$ be a 3-fully dualisable symmetric monoidal 3-category. The cobordism hypothesis implies $O(3)$ should act on $k(\mathcal{C}^{fd})$.

$O(3) = SO(3) \rtimes \mathbb{Z}/_2$. Previously we saw that 1-dualisability gave the $\mathbb{Z}/_2$ action. 2-dualisability gave furthermore an $SO(2)$ action, which we could explicitly understand via the Serre automorphism $S: id_{k(\mathcal{C}^{fd})} \xrightarrow{\simeq} id_{k(\mathcal{C}^{fd})}$, & a quadratic invariant $q(S): id_{id_{k(\mathcal{C}^{fd})}} \xrightarrow{\simeq} id_{id_{k(\mathcal{C}^{fd})}}$. The $SO(2)$-structure was an iso $q(S) \cong id_{id_{id_{k(\mathcal{C}^{fd})}}}$.

**Exercise:**
Show $S_x^{-1} =$  : $ev^L$ <u>left</u> adjoint to $ev$.

**Observe:** An $SO(3)$ induces an $SO(2)$ action. Let's compare $SO(2)$ & $SO(3)$

| | $SO(2)$ | $SO(3)$ |
|---|---|---|
| $\pi_1$ | $\mathbb{Z}$ | $\mathbb{Z}/_2$ |
| $\pi_2$ | 0 | 0 |
| $\pi_3$ | 0 | $\mathbb{Z}$ |

So expect the Serre to be order two. Why is this? It follows from an extraordinary fact:

**Lemma:**
Let $\mathcal{C}$ be a sym. mon. 3-category, & $f: x \to y$ a 1-morphism. Suppose that $f$ admits a right dual

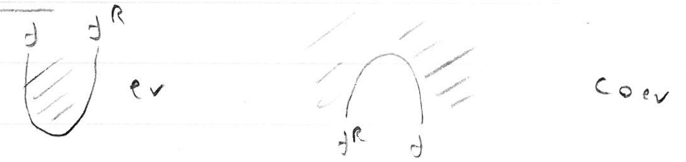
So we have $(f, f^R, ev: f \circ f^R \to 1_y, coev: 1_x \to f^R \circ f)$.
Suppose <u>further</u> that ev & coev themselves admit <u>left</u> duals. Then $f^R$ is actually also a <u>left</u> dual: $(f^R, f, coev^L, ev^L)$ exhibits $f^R$ as left dual to $f$ canonically.

## <u>Corollary</u>:

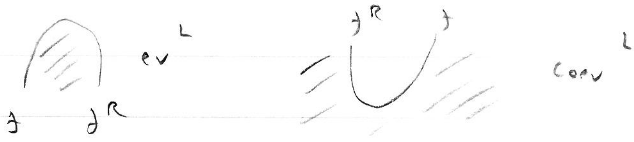
In our 3-fully dualisable setting, there is a canonical natural isomorphism $R: S^2 \to id_{id}$ (<u>Radford</u> isomorphism). This is immediate from the exercise.

## <u>Proof sketch</u>:

Draw

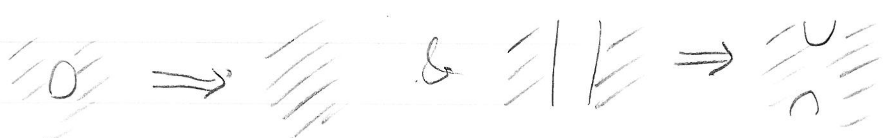



and

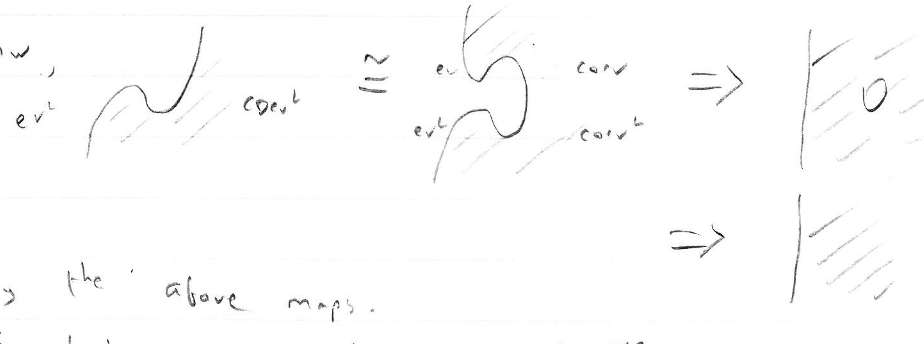



We want zig-zag conditions for these latter 2-morphisms.
One has maps $ev^L \circ ev \to id$, $id \to ev \circ ev^L$, which we draw


& similarly for coev



Now,




by the above maps.
Similarly, one constructs an inverse. □

**Theorem** (D-SP-S)

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal 3-category. Then to give an $SO(3)$ action is to give

- Serre $S : id_{\mathcal{C}} \xrightarrow{\simeq} id_{\mathcal{C}}$
- $\sigma : q(S) \xrightarrow{\simeq} id_{id_{id_{\mathcal{C}}}}$
- Radford $R : S^2 \xrightarrow{\simeq} id_{id}$

As a consequence, we can completely understand the $SO(3)$ action on $k(\mathcal{C}^{fd})$.

In particular, we can deduce the $\pi_3$ from this data. The generator will be some

$$a : id_{id_{id_{k(\mathcal{C})}}} \longrightarrow id_{id_{id_{k(\mathcal{C})}}}$$

Get this by looking at

$$q(R) : \underset{\substack{\cong \\ q(S)^4}}{q(S^2)} \longrightarrow id_{id_{id}}$$

$q(S)^4$ as $q$ is quadratic.

So say $a : id_{id_{id}} \xrightarrow{(\sigma^{-1})^{\otimes 4}} q(S)^4 \xrightarrow{q(R)} id_{id_{id}}$

**Proof sketch:**

Want a map completing the diagram:

$$\begin{tikzcd}
S^2 \arrow[r] \arrow[d] & BSO(2) \arrow[d] \arrow[ddr, "{(S, \sigma)}"] & \\
BOrpo(3) \arrow[r] \arrow[drr, bend right, "{(S, R)}"'] & BSO(3) \arrow[dr, dashed] & \\
 & & BAut^{\circ}(\mathcal{C})
\end{tikzcd}$$

$BOrpo(3)$ is the htpy fibre of $BSO(3) \xrightarrow{p_1} K(\mathbb{Z}, 4)$, the first Pontryagin class. It classifies "$p_1$-structures". $\pi_1\, Orpo(3) = \mathbb{Z}/2$

$\pi_2\, Orpo(3) = \Gamma(\mathbb{Z}/2) \cong \mathbb{Z}/4$.

This square is a push-out of 4-types. $\square$

## Application:

Definition: Fix a field $k$.

A fusion category is a monoidal $k$-linear abelian category such that

- It is semisimple, with finite-dimensional homs & finitely many iso classes of simple objects.
- It is rigid, i.e. every object has both left and right duals.

## Examples:

- Representation categories of finite quantum groups or semisimple finite-dimensional Hopf algebras
- Representation categories of Loop groups (positive energy), hence CFT.
- von Neumann algebras $A \subseteq B$ leads to a basic invariant, essentially a fusion category.

**Theorem** (Etingof, N , Ostrik)

If $(F, \otimes)$ is a fusion category, there's a monoidal endofunctor $F \longrightarrow F$ via $x \mapsto x^{****}$. Then it is canonical isomorphic to the identity.

The usual proof goes as follows:

$F \simeq Rep(H)$, for $H$ a weak Hopf algebra non-canonically. One shows $H$ satisfies an analogue of "Radfords $S^4$ formula", & deduce facts about $Rep(H)$ from it.

$S$ here is an antipode in $H$.

There's a symmetric monoidal 3-category whose objects are fusion categories, 1-morphisms are bimodule

categories, 2-morphisms are functors &
3-morphisms are natural transformations.
(Categorification of the Morita category of algebras).

<u>Theorem</u> (D-Sp-S)
This category is 3-fully dualisable, &
the Serre is given by the bimodule category

$$S_F = {}_F F_{F^{**}}$$

<u>Corollary</u>.
The quadruple dual is equivalent to id
(by the Radford isomorphism).